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A. Sasane

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# Hankel Norm Approximation for Infinite-Dimensional Systems

With 19 Figures



Springer

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# Preface

This book is aimed primarily at system theorists working on the approximation of systems described in terms of linear partial differential equations by systems described in terms of linear ordinary differential equations. However, it will be of interest also to functional analysts with an interest in the classical problems of interpolation theory of analytic functions.

Model reduction is an important engineering problem in which one aims to replace an elaborate model by a simpler model without undue loss of accuracy. The accuracy can be mathematically measured in several possible norms and the Hankel norm is one such. The Hankel norm gives a meaningful notion of distance between two linear systems: roughly speaking, it is the induced norm of the operator that maps past inputs to future outputs. It turns out that the engineering problem of model reduction in the Hankel norm is closely related to the mathematical problem of finding solutions to the sub-optimal Nehari-Takagi problem, which we call “the sub-optimal Hankel norm approximation problem” in this book. Although the existence of a solution to the sub-optimal Hankel norm approximation problem has been known since the 1970’s, in this book, we give explicit solutions and, in particular, we give new formulae for several large classes of infinite-dimensional systems.

The approach taken in this book is as follows. First we give a “frequency domain” solution to the sub-optimal Hankel norm approximation problem. We start with a complex matrix-valued function  $G$  defined on the imaginary axis satisfying certain assumptions. In particular, we demand the existence of a solution to a certain  $J$ -spectral factorization problem. We then give a solution to the sub-optimal Hankel norm approximation problem in terms of the  $J$ -spectral factor. Furthermore, we give a parameterization of all solutions in terms of the  $J$ -spectral factor, the parameterizing set being the unit ball in a certain Hardy space. In this manner we give purely “frequency domain” solutions to the sub-optimal Hankel norm approximation problem in Chapter 4 of this book.

Subsequently, we give “state-space” solutions to the sub-optimal Hankel norm approximation problem for important classes of infinite-dimensional linear systems. In Chapter 5 we consider the case where  $G$  is the transfer function of a well-posed linear system given by a triple of operators  $(A, B, C)$ . Under certain assumptions we solve the sub-optimal Hankel norm approximation problem for  $G = C(sI - A)^{-1}B$  by constructing a  $J$ -spectral factor in terms of the system parameters  $(A, B, C)$  and verifying that this constructed  $J$ -spectral factor satisfies the assumptions demanded in Chapter 4. In this manner, we obtain “state-space” solutions to the sub-optimal Hankel norm approximation problem for two important classes of well-posed linear systems: the smooth Pritchard-Salamon class of exponentially stable infinite-dimensional systems and the class of exponentially stable analytic systems.

Finally, we also solve the sub-optimal Hankel norm approximation problem for certain classes of infinite-dimensional systems with a non-exponentially stable semigroup and for regular linear systems.

This book is a slightly adapted and extended version of my Ph.D. thesis. It is the outcome of a period of four years at the Department of Mathematics of the University of Groningen, The Netherlands. My advisor during this period was Professor Ruth Curtain, and the results in this thesis are the product of cooperation with her. I would like to express my deepest gratitude for this. Finally, I wish to thank my family for everything.

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# Chapter 1

## Introduction

Systems and control theory is a subject in which beautiful and deep mathematical theorems have exactly matched the needs of a vital branch of technology. This conjunction of mathematics and engineering has been enriching for both fields. In this book we present a solution to the sub-optimal Hankel norm approximation problem which has been studied by both mathematicians and engineers. In this chapter, among other things, we explain the problem and the relevance of its solution to the important engineering problem of model reduction, in which the aim is to replace a high order (possibly infinite-dimensional) system by a low order model, without undue loss of accuracy. Finally we will give a brief outline of the chapters in this book. But first we give a brief introduction to the history of the problem, based on Young [93].

In 1957, Zeev Nehari [56] studied the following natural problem. Suppose we are given a sequence  $\{a_n\}_{n \geq 0}$  of complex numbers. Under what conditions does the infinite *Hankel matrix*<sup>1</sup>

$$H = \begin{bmatrix} a_0 & a_1 & a_2 & \dots \\ a_1 & a_2 & a_3 & \dots \\ a_2 & a_3 & a_4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (1.1)$$

define a bounded linear operator on  $\ell_2(\mathbb{N})$ ?

He found a striking answer: a necessary and sufficient condition is that there should exist an essentially bounded function  $\varphi$  on the unit circle  $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$  (parameterized as  $\{e^{i\theta} \mid 0 \leq \theta \leq 2\pi\}$ ) such that

$$\text{the } n^{\text{th}} \text{ Fourier coefficient } \hat{\varphi}(n) = a_n, \text{ for all } n \geq 0. \quad (1.2)$$

---

<sup>1</sup>That is, a matrix  $\{c_{k1}\}_{k,1 \geq 0}$ , where  $c_{k1}$  depends only on  $k+1$  and so  $c_{k1} = a_{k+1}$  for some sequence  $\{a_n\}_{n \geq 0}$ .



Hermann Hankel (1839-1873).

(Courtesy of <http://www-groups.dcs.st-andrews.ac.uk/~history>)

Furthermore, when (1.2) is satisfied,

$$\|H\|_{\mathcal{L}(\ell_2(\mathbb{N}))} = \text{infimum of the } L_\infty \text{ - norms of all such } \varphi. \quad (1.3)$$

**Example 1.0.1** If

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

the *Hilbert matrix*, Schur proved that  $\|H\|_{\mathcal{L}(\ell_2(\mathbb{N}))} = \pi$ . By taking  $\varphi(e^{i\theta}) = i(\pi - \theta)e^{i\theta}$ , we have a function with  $L_\infty$ -norm at most  $\pi$ , and

$$\begin{aligned} \hat{\varphi}(\mathbf{n}) &= \frac{1}{2\pi} \int_0^{2\pi} i(\pi - \theta)e^{i\theta} e^{-i\mathbf{n}\theta} d\theta \\ &= \left[ \frac{-i(\pi - \theta)e^{-i(\mathbf{n}+1)\theta}}{2\pi(\mathbf{n}+1)} \right]_0^{2\pi} - \frac{i}{2\pi(\mathbf{n}+1)} \int_0^{2\pi} e^{-i(\mathbf{n}+1)\theta} d\theta \\ &= \frac{1}{\mathbf{n}+1}, \end{aligned}$$

for all  $\mathbf{n} \geq 0$ . ◇

We can reformulate the above result as follows. Let

$$G(z) = \sum_{\mathbf{n}=-\infty}^{\infty} g_{\mathbf{n}} z^{\mathbf{n}} \quad \text{and} \quad g_{\mathbf{n}} = a_{\mathbf{n}} \quad \text{for all } \mathbf{n} \geq 0.$$

In (1.3), the right hand side is the infimum of  $\|\varphi\|_\infty$  over all  $L_\infty$ -functions  $\varphi$  on the circle such that

$$\hat{\varphi}(\mathbf{n}) = a_n, \quad \mathbf{n} \geq 0.$$

This class of  $\varphi$  is exactly the coset  $G + H_\infty(\mathbb{D})^\perp$  in  $L_\infty(\mathbb{T})$ , where  $H_\infty(\mathbb{D})^\perp$  is the subspace (of  $L_\infty(\mathbb{T})$ ) of functions whose nonnegative Fourier coefficients vanish, which is the space of functions which are analytic and bounded in  $\{z \in \mathbb{C} \mid |z| > 1\}$ . Hence

$$\begin{aligned} \|H\|_{\mathcal{L}(\ell_2(\mathbb{N}))} &= \inf_{K \in H_\infty(\mathbb{D})^\perp} \|G + K\|_\infty \\ &= \text{dist}_{L_\infty(\mathbb{T})}(G, H_\infty(\mathbb{D})^\perp). \end{aligned}$$

So Nehari's theorem gives us an expression for the distance of a bounded function on the circle from  $H_\infty(\mathbb{D})^\perp$  as the norm of a Hankel matrix, and so it gives a lower bound on the error if we are finding an approximation in  $H_\infty(\mathbb{D})^\perp$  of a given function in  $L_\infty(\Gamma)$ .

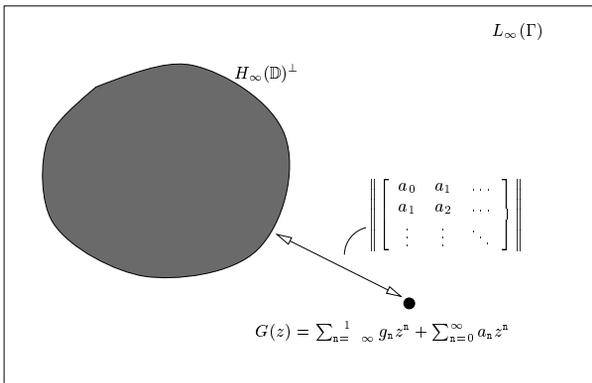


Figure 1.1: Nehari's theorem.

More detailed information relating to this type of  $L_\infty$ -approximation on the basis of Hankel operator theory was subsequently obtained by V.M. Adamjan, D.Z. Arov and M.G. Krein [2], D. Sarason [74] and S.R. Treil [82]. An excellent introduction to the subject is Young [93], upon which this chapter is based, and a survey of results can be found in Nikol'skii [57]. The fact that results on such  $L_\infty$ -approximation problems were relevant to the preoccupations of some engineers was realized by a number of mathematicians in the seventies. This stimulated further research and the birth of an intensive period of research and interaction by both engineers and mathematicians—the  $H_\infty$ -era of the 80's (see Francis [32]).

## 1.1 The sub-optimal Nehari problem

A reformulation of the question answered<sup>2</sup> by Nehari is the following: Given a bounded  $\mathfrak{p} \times \mathfrak{m}$  matrix-valued function  $G$  on the imaginary axis, find the distance of  $G$  from  $H_\infty(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})^\perp$  in the  $L_\infty$ -norm, where  $H_\infty(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})^\perp$  denotes the space of  $\mathfrak{p} \times \mathfrak{m}$  matrix-valued functions which are bounded and analytic in the open left half-plane. That is, find

$$\begin{aligned} \text{dist}_{L_\infty} \left( G, H_\infty(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})^\perp \right) &= \inf_{K(-) \in H_\infty(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})} \|G(i \cdot) + K(i \cdot)\|_\infty \\ &= \inf_{K(-) \in H_\infty(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})} \text{esssup}_{\omega \in \mathbb{R}} \|G(i\omega) + K(i\omega)\|, \end{aligned}$$

where we use the notation  $H_\infty(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$  for the space of  $\mathfrak{p} \times \mathfrak{m}$  matrix-valued functions which are bounded and analytic in the open right half-plane. Another way of phrasing the question is to say that we are given a point  $G$  in the Banach space  $L_\infty$  and we seek the closest point or points in the closed subspace  $H_\infty(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})^\perp$ . It is perhaps not surprising that in any Banach space other than a Hilbert space, best approximation problems are usually difficult and one does not expect to solve them exactly. Fortunately for control theory, the Nehari problem is one of the rare cases which does admit a precise solution.

Let  $H_2(\mathbb{C}^{\mathfrak{k}})$  denote the set of all analytic functions  $f : \mathbb{C}_0^+ \rightarrow \mathbb{C}^{\mathfrak{k}}$  such that

$$\|f\|_2 := \sup_{\zeta > 0} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \|f(\zeta + i\omega)\|^2 d\omega \right)^{\frac{1}{2}} < \infty.$$

For  $G \in L_\infty(i\mathbb{R}, \mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$  we define the *Hankel operator with symbol  $G$* , denoted by  $H_G$ , acting from  $H_2(\mathbb{C}^{\mathfrak{m}})$  to  $H_2(\mathbb{C}^{\mathfrak{p}})$ , as follows:

$$H_G f = \Pi(M_G f_-) \text{ for } f \in H_2(\mathbb{C}^{\mathfrak{m}}),$$

where  $M_G$  is the multiplication map on  $L_2(i\mathbb{R}, \mathbb{C}^{\mathfrak{m}})$  induced by  $G$ ,  $\Pi$  is the orthogonal projection operator from  $L_2(i\mathbb{R}, \mathbb{C}^{\mathfrak{p}})$  onto  $H_2(\mathbb{C}^{\mathfrak{p}})$  and  $f_-(s) := f(-s)$ . The operator  $H_G$  is related to the infinite Hankel matrix  $H$  in (1.1):  $H_G$  has a matrix of that form with respect to a suitable orthonormal basis. Nehari's theorem says that for any  $G \in L_\infty(i\mathbb{R}, \mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$ ,

$$\text{dist}_{L_\infty} \left( G, H_\infty(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})^\perp \right) = \|H_G\|.$$

One is of course interested not only in the value of the distance, but also in characterizing the set of  $K$ 's for which the distance is attained. In the case of rational transfer functions there are explicit solutions for  $K$  (see Glover [34]). However, for control applications, the focus is on the sub-optimal Nehari problem. The *sub-optimal Nehari problem* is the following:

<sup>2</sup>The matrix case was settled by Adamjan, Arov and Krein in [1].

Let  $G(i \cdot) \in L_\infty(\mathbb{R}, \mathbb{C}^{\mathbf{p} \times \mathbf{m}})$ .  
 If  $\sigma > \|H_G\|$ , then find  $K(-\cdot) \in H_\infty(\mathbb{C}^{\mathbf{p} \times \mathbf{m}})$  such that  
 $\|G(i \cdot) + K(i \cdot)\|_\infty \leq \sigma$ .

The solutions  $K$  to the sub-optimal Nehari problem are the key to the design of controllers which maximize robustness with respect to uncertainty or minimize sensitivity to disturbances of sensors (see McFarlane and Glover [54] and Georgiou and Smith [33]). For the purpose of “model reduction”, a notion to be explained later in this chapter, one considers a natural generalization of the Nehari problem, and this problem turns out to be much harder.

## 1.2 The sub-optimal Hankel norm approximation problem

Let  $H_{\infty,1}(\mathbb{C}^{\mathbf{p} \times \mathbf{m}})$  denote the space of all  $\mathbf{p} \times \mathbf{m}$  matrix-valued functions  $K$  of a complex variable defined in the right half-plane such that  $K = G_* + F$ , where  $F$  is an element in  $H_\infty(\mathbb{C}^{\mathbf{p} \times \mathbf{m}})$  and  $G_*$  is the transfer function of a finite-dimensional system with MacMillan degree<sup>3</sup> at most 1, with all its poles in the open right half-plane.  $H_{\infty,1}(\mathbb{C}^{\mathbf{p} \times \mathbf{m}})$  is a subset of  $L_\infty(i\mathbb{R}, \mathbb{C}^{\mathbf{p} \times \mathbf{m}})$ .

Now we recall the notion of singular values of a bounded linear operator from a Hilbert space  $\mathcal{H}_1$  to a Hilbert space  $\mathcal{H}_2$ . For  $\mathbf{k} \in \{1, 2, \dots\}$  the  $\mathbf{k}^{\text{th}}$  *singular value* of an operator  $\mathbb{H} \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  (denoted by  $\sigma_{\mathbf{k}}(\mathbb{H})$ ) is defined to be the distance with respect to the norm in  $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  of  $\mathbb{H}$  from the set of operators in  $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  of rank at most  $\mathbf{k} - 1$ . Thus  $\sigma_1(\mathbb{H}) = \|\mathbb{H}\|$ , and  $\sigma_1(\mathbb{H}) \geq \sigma_2(\mathbb{H}) \geq \sigma_3(\mathbb{H}) \geq \dots \geq 0$ . If  $\mathbb{H}$  is compact, then  $\mathbb{H}^*\mathbb{H}$  is compact and nonnegative, and so the spectrum of  $\mathbb{H}^*\mathbb{H}$  consists of a pure point spectrum with countably many nonnegative eigenvalues. The square roots of these eigenvalues are then the singular values of  $\mathbb{H}$ .

If the Hankel operator with symbol  $G \in L_\infty(i\mathbb{R}, \mathbb{C}^{\mathbf{p} \times \mathbf{m}})$  is compact, we sometimes denote the singular values of  $H_G$ ,  $\sigma_{\mathbf{k}}(H_G)$ , simply by  $\sigma_{\mathbf{k}}(G)$ . The  $\sigma_{\mathbf{k}}(G)$ 's are then referred to as the *Hankel singular values of  $G$* .

---

<sup>3</sup>The *MacMillan degree* of a proper rational function  $G$  is defined to be the minimal state dimension of all possible  $(A, B, C, D)$ 's that realize  $G$ , that is

$$\text{MacMillan degree of } G = \min_{\substack{A \in \mathbb{C}^{\mathbf{n}}, \\ B \in \mathbb{C}^{\mathbf{m} \times \mathbf{n}}, \\ C \in \mathbb{C}^{\mathbf{n} \times \mathbf{p}}, \\ D \in \mathbb{C}^{\mathbf{p} \times \mathbf{m}}}} \{\mathbf{n} \mid G(s) = C(sI - A)^{-1}B + D\}.$$

The following theorem<sup>4</sup> of Adamjan, Arov and Krein [1] is a natural extension of Nehari's theorem:

**Theorem 1.2.1** For any  $G \in L_\infty(i\mathbb{R}, \mathbb{C}^{\mathbf{p} \times \mathbf{m}})$ ,

$$\inf_{K(\cdot) \in H_{\infty,1}(\mathbb{C}^{\mathbf{p} \times \mathbf{m}})} \|G(i\cdot) + K(i\cdot)\|_\infty = \sigma_{1+1}(G).$$

The Nehari theorem corresponds to the case  $\mathbf{1} = 0$ . In Theorem 1.2.1, if  $K(\cdot) = G_*(\cdot) + F(\cdot)$ , where  $G_*(\cdot)$  is the rational transfer function of a finite-dimensional system with MacMillan degree at most  $\mathbf{1}$  and with all its poles in the open right half-plane, and  $F(\cdot) \in H_\infty(\mathbb{C}^{\mathbf{p} \times \mathbf{m}})$ , then  $-G_*(\cdot)$  is said to be an *optimal Hankel norm approximant* of  $G$ . The *sub-optimal Hankel norm approximation problem* is the following:

$$\begin{aligned} & \text{Let } G(i\cdot) \in L_\infty(\mathbb{R}, \mathbb{C}^{\mathbf{p} \times \mathbf{m}}). \\ & \text{If } \sigma_{1+1} < \sigma < \sigma_{\mathbf{1}}, \text{ then find } K(\cdot) \in H_{\infty,1}(\mathbb{C}^{\mathbf{p} \times \mathbf{m}}) \\ & \text{such that } \|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma. \end{aligned}$$

$K$  is then called a *solution* of the sub-optimal Hankel norm approximation problem.

## 1.3 The model reduction problem

An elaborate model of a physical system can sometimes be replaced by a simpler one, which is easier to analyze and use, without a significant loss of accuracy. In this section we shall see how a solution to the sub-optimal Hankel norm approximation problem gives rise to a finite-dimensional reduced model to the original infinite-dimensional system.

### 1.3.1 The Hankel norm of systems

In practice, many physical systems and engineering devices are modelled by mathematical systems of the type

$$\left. \begin{aligned} x(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \right\} t \geq 0,$$

---

<sup>4</sup>the matrix case was proved in Kung and Lin [50].

where  $A$  is the infinitesimal generator of a strongly continuous semigroup  $\{T(t)\}_{t \geq 0}$  on the Hilbert space  $X$ ,  $B \in \mathcal{L}(\mathbb{C}^m, X)$  and  $C \in \mathcal{L}(X, \mathbb{C}^p)$ .<sup>5</sup>

If  $A$  is *exponentially stable* (that is, there exists an  $\epsilon < 0$  and a  $M > 0$  such that  $\|T(t)\| \leq Me^{-\epsilon t}$  for all  $t \geq 0$ ), then the *impulse response*  $h(\cdot) = CT(\cdot)B$  is an element belonging to  $L_1([0, \infty), \mathbb{C}^{p \times m})$ , and  $h$  induces a compact *time-domain Hankel operator*  $\Gamma_h \in \mathcal{L}(L_2([0, \infty), \mathbb{C}^m), L_2([0, \infty), \mathbb{C}^p))$ , defined as follows: given  $u \in L_2([0, \infty), \mathbb{C}^m)$ ,

$$(\Gamma_h u)(t) = \int_0^\infty h(t + \tau)u(\tau)d\tau \text{ for all } t \geq 0.$$

The impulse response  $h$  has the Laplace transform  $G(s) = C(sI - A)^{-1}B$  with  $G(\cdot) : i\mathbb{R} \rightarrow \mathbb{C}^{p \times m} \in L_\infty(i\mathbb{R}, \mathbb{C}^{p \times m})$ . This induces a compact Hankel operator  $H_G$  with symbol  $G$ , which is related to  $\Gamma_h$  via Laplace transformation:

$$H_G \hat{f}(s) = \left( \widehat{\Gamma_h f} \right)(s) \text{ for all } s \text{ in the open right half-plane,}$$

and  $\|\Gamma_h\| = \|H_G\| \leq \|G\|_\infty \leq \|h\|_{L_1}$ . If  $u_* \in L_2((-\infty, 0], \mathbb{C}^m)$ , then  $u(\cdot) := u_*(-\cdot) \in L_2([0, \infty), \mathbb{C}^m)$ , and we have for  $t \geq 0$

$$(\Gamma_h u)(t) = \int_0^\infty CT(t + \tau)u(\tau)d\tau = \int_{-\infty}^0 CT(t - \tau)u_*(\tau)d\tau.$$

Roughly speaking, if we run the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu_*(t), \\ y(t) &= Cx(t), \end{aligned}$$

from time  $-\infty$  to 0 with input function  $u_*$  and “initial state  $x(-\infty) = 0$ ”, and then cease to input anything and observe the output  $y(\cdot)$  from time 0 onwards, then  $(\Gamma_h u)(t) = y(t)$  for all  $t \geq 0$  (see Figure 1.2).

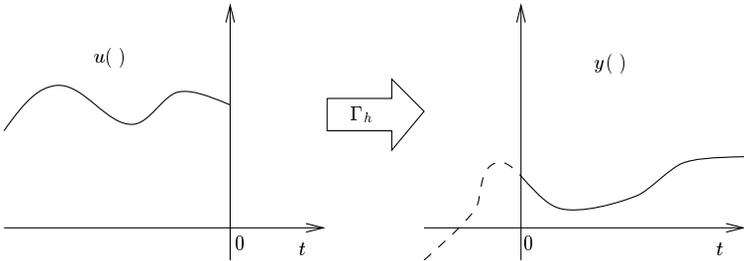


Figure 1.2: The time-domain Hankel operator.

<sup>5</sup>More generally, the input space  $\mathbb{C}^m$  and the output space  $\mathbb{C}^p$  could be Hilbert spaces  $U$  and  $Y$ , respectively, and  $B$  and  $C$  could be *unbounded*.

In other words,  $\Gamma_h$  is the mapping from past inputs to future outputs. The *Hankel norm of a system with transfer function  $G$*  (sometimes denoted by  $\|G\|_H$ ) is defined to be  $\|H_G\|$ . It is now evident that this is a physically meaningful notion of closeness of systems. To say that  $G_*$  is close to  $G$  in the Hankel norm means that the corresponding systems acting on a past input signal give future outputs which are close to each other.

### 1.3.2 The model reduction problem

In some areas of engineering, linear infinite-dimensional state-space models of dynamic systems are derived and it is desirable to replace them by finite-dimensional systems without incurring too much error. This would make the subsequent control system design computationally less demanding and possibly numerically more reliable. A wide variety of methods for model reduction have been proposed over the years; here we list a few:

1. Balanced model reduction: See for example Moore [55] and Pernebo and Silverman [64] for the finite-dimensional case and Curtain and Glover [36], Ober [58], Young [92] for the infinite-dimensional case.
2. Approximation of systems with  $h \in L_1 \cap L_2$  with a nuclear Hankel operator: See Curtain, Glover and Partington [37].
3. Other miscellaneous techniques: See Curtain [12], Glover, Lam and Partington [38], [39], Gu, Khargonekar and Lee [43].

However, in this book, we consider the *model reduction problem with respect to the Hankel norm*:

Given a stable  $G$  and a nonnegative integer  $l$ ,  
find a stable rational  $G_*$  of MacMillan degree at most  $l$   
such that  $\|G - G_*\|_H$  is small.

We explain the relation of this question to the problem of sub-optimal Hankel norm approximation in the next section.

### 1.3.3 Hankel norm approximation and model reduction

Suppose that  $G \in L_\infty(i\mathbb{R}, \mathbb{C}^{p \times m})$  has a compact Hankel operator and that if  $\sigma_{l+1} < \sigma < \sigma_l$ ,  $K(-\cdot) \in H_{\infty,1}(\mathbb{C}^{p \times m})$  is a solution to the sub-optimal Hankel norm approximation problem, that is,

$$\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma.$$

Let  $K(\cdot) = G_*(\cdot) + F(\cdot)$ , where  $G_*(\cdot)$  is the rational transfer function of a finite-dimensional system with MacMillan degree at most 1 and with all its poles in the open right half-plane, and  $F(\cdot) \in H_\infty(\mathbb{C}^{\mathbb{P}^{\times m}})$ . Then  $-G_*(\cdot)$  (called a *sub-optimal Hankel norm approximant*) is a reduced order model of the original system  $G$  in the Hankel norm, with error at most equal to  $\sigma$ . Indeed, we have

$$\|G + G_*\|_H = \|G + G_* + F\|_H = \|G + K\|_H \leq \|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma. \quad (1.4)$$

It is not clear that closeness in the Hankel norm is an appropriate criterion for a reduced order model. Most robust control designs are robust with respect to the stronger  $L_\infty$ -norm:  $\|G\|_H \leq \|G\|_\infty$ . However, in practice the Hankel norm approximants performed well. The explanation of this fortunate fact was provided by Glover in his seminal paper [34], which was generalized in [37].

In Glover et al. [37] it was shown that if  $h \in L_1([0, \infty), \mathbb{C}^{\mathbb{P}^{\times m}}) \cap L_2([0, \infty), \mathbb{C}^{\mathbb{P}^{\times m}})$ , and if  $\sum_{k=1}^{\infty} \sigma_k < \infty$ , then for a given integer 1, there exists an **optimal** Hankel-norm approximation  $G_0$  of order 1, such that

$$\|G - G_0\|_H \leq \|G - G_0\|_\infty \leq \sum_{k=1+1}^{\infty} \sigma_k(G).$$

It is very commonly the case that the Hankel operator has a few sizeable singular values and the remaining tail away very quickly to zero. In such a case, the right hand side can be made very small, and one is assured that an optimal Hankel norm approximant is also good with respect to the  $L_\infty$ -norm. This norm is extremely desirable because there exist many robustness theories with respect to this norm.

In this book, we only consider the sub-optimal Hankel-norm approximation problem. In (1.4) above we showed how this yields a sub-optimal Hankel norm approximant  $-G_*(\cdot)$  which is close in the Hankel norm. However, for control applications, we prefer closeness in the  $L_\infty$ -norm. Since we were unable to find any results on this, even in the finite-dimensional literature, we prove (in Section 3.3) that the error bound in Corollary 7.5 (see page 69, Partington [62]) for an optimal Hankel norm approximant also holds for a sub-optimal Hankel norm approximant: If  $h \in L_1([0, \infty), \mathbb{C}^{\mathbb{P}^{\times m}})$  and  $\sum_{k=1}^{\infty} \sigma_k < \infty$ , then any **sub-optimal** Hankel norm approximant  $G_*$  as in (1.4) of MacMillan degree 1 satisfies

$$\|G(i\cdot) + G_*(i\cdot)\|_\infty \leq 41 \sigma_1(G) + 2 \sum_{k=1+1}^{\infty} \sigma_k(G).$$

Moreover we will show that the error bound

$$E_1 := 41 \sigma_1(G) + 2 \sum_{k=1+1}^{\infty} \sigma_k(G)$$

tends to zero as  $1 \rightarrow \infty$ . So the sub-optimal Hankel norm approximant is also a good approximation of  $G$  in the  $L_\infty$ -norm.

While the natural formulation of the sub-optimal Hankel norm approximation problem is in terms of the transfer function, most systems in control applications are modelled in state-space form in terms of parameters  $A$ ,  $B$ ,  $C$ . The Hankel norm approximation problem has been solved for the rational case in Glover [34], Ball and Ran [6] and for the special class of Pritchard-Salamon infinite-dimensional systems in Curtain and Ran [17]. Our aim in this book has been two-fold:

1. To obtain sufficient conditions for the existence of a solution to the sub-optimal Hankel norm approximation problem in terms of a solution to a key  $J$ -spectral factorization problem. This allows a complete parameterization of all solutions in terms of the spectral factor (see Chapter 4).
2. For certain classes of well-posed linear infinite-dimensional systems, we use the frequency domain results established in Chapter 4 to obtain explicit solutions to the sub-optimal Hankel norm approximation problem in terms of the state-space parameters  $A$ ,  $B$ ,  $C$ . The classes considered are exponentially stable, smooth Pritchard-Salamon systems (Section 5.1 of Chapter 5), exponentially stable analytic systems (Section 5.2 of Chapter 5), non-exponentially stable systems (Chapter 6) and regular linear systems (Chapter 7).

The above strategy is reminiscent of the strategy in Ball and Ran [6], Ran [67] and Curtain and Ran [17] with one important difference. In these three references, the starting point was to quote a result (Theorem 4.4.1 of Section 4.4 in Chapter 4) which was claimed to be a corollary of the results in Ball and Helton [5]. Since it is not directly clear to non experts and certainly not to the engineering community how this result follows from [5], we found it worthwhile to derive a simpler frequency domain result from first principles using elementary mathematics in Chapter 4. We remark that an analogous strategy has been followed earlier to solve the special case of the sub-optimal Nehari problem in Curtain and Zwart [27], [26], Curtain and Oostveen [16] and Curtain and Ichikawa [14].

## 1.4 Organization of the book

In this section, we give an overview of the results in this book.

**Chapter 2:** We first define well-posed linear systems, which forms the general framework for studying infinite-dimensional systems with a state space representation. We recall a number of definitions concerning this class of systems

which are relevant to the remainder of this book. Next we discuss two important classes of well-posed linear systems:

1. the Pritchard-Salamon class, and
2. the analytic class,

which are central in this book. Some of their properties, which will be used in Chapter 5 are listed. Finally, in the last section of this chapter we will introduce several frequency domain spaces and prove a few elementary lemmas which will be used later.

**Chapter 3:** In the theory of approximation of infinite-dimensional systems, the nuclearity and compactness properties of the Hankel operator play an essential role. In this chapter we investigate the relationships between the exponential (or strong) stability of certain classes of regular linear systems and the compactness and nuclearity properties of the Hankel operator by means of considering illustrative examples. New sufficient conditions for nuclearity are given for exponentially stable regular linear systems with an analytic semigroup. In the last section of this chapter we will derive a bound on the  $L_\infty$ -error of a sub-optimal Hankel norm approximant of a system with a nuclear Hankel operator.

**Chapter 4:** In this chapter, first we prove a few useful consequences of the main (frequency domain) assumptions S1-5 in this chapter, and subsequently, under these assumptions, we give a characterization of all solutions to the sub-optimal Hankel norm approximation problem. These derivations are new.

Later in Chapter 5, we will show that for the Pritchard-Salamon class and the analytic class, there do exist  $J$ -spectral factors for each class satisfying the assumptions in this chapter, hence solving the sub-optimal Hankel norm approximation problem for these classes of infinite-dimensional systems. We will give an explicit formula for a  $J$ -spectral factor  $\Lambda$  in terms of the state-space parameters  $(A, B, C)$  of the original system.

**Chapter 5:** In Section 5.1, the sub-optimal Hankel norm approximation problem is solved for the smooth Pritchard-Salamon class of exponentially stable infinite-dimensional systems with finite-dimensional input and output spaces by using the results from Section 2.3 of Chapter 2 and Chapter 4: we give a formula for a  $J$ -spectral factor  $\Lambda$  in terms of the state-space parameters, and check that it has all the properties required in Chapter 4. These results were obtained earlier in Curtain and Ran [17] by appealing to the Ball and Helton result [5].

Next, in Section 5.1, the sub-optimal Hankel norm approximation problem is solved for the analytic class of exponentially stable infinite-dimensional systems with finite-dimensional input and output spaces by using the results

from Section 2.4 of Chapter 2 and Chapter 4. Again, we give a formula for a  $J$ -spectral factor  $\Lambda$  in terms of the state-space parameters, and check that it has all the properties required in Chapter 4. The results of this chapter are new and include the results on the Nehari problem in Curtain and Ichikawa [14] as a special case.

**Chapter 6:** In the previous chapters the sub-optimal Hankel norm approximation problem was solved for various classes of infinite-dimensional systems under the assumption that  $A$  generates an *exponentially* stable strongly continuous semigroup. In this chapter, we will solve the sub-optimal Hankel norm approximation problem for non-exponentially stable infinite-dimensional systems in terms of a solution to the sub-optimal Hankel norm approximation problem for an exponentially stable system. The exponentially stable system is obtained by shifting the generator of the semigroup of the original system. The results in this chapter are new.

**Chapter 7:** In this chapter, we will solve the sub-optimal Hankel norm approximation problem for regular linear systems with generating operators  $(A, B, C)$  satisfying some mild assumptions. Furthermore, in the special case of the sub-optimal Nehari problem, we will also give an explicit parameterization of all solutions. The class of regular linear systems is a large class of infinite-dimensional state-linear systems allowing for unbounded inputs and outputs; it includes the Pritchard-Salamon class and the analytic class. But the solution obtained in this chapter is different from the form in the preceding chapters.

# Chapter 2

## Classes of well-posed linear systems

In this chapter we recall the notion of well-posed linear systems, which forms the general framework for studying infinite-dimensional systems in a state space context. In particular, we introduce two important classes of well-posed linear systems which are central in this book and list some of their properties, which will be used in the sequel. Figure 2.1 shows the hierarchy of the various classes of systems that we will encounter in this chapter. Finally, in the last section of this chapter we will introduce several frequency domain spaces and prove a few elementary lemmas which will be used in what follows.

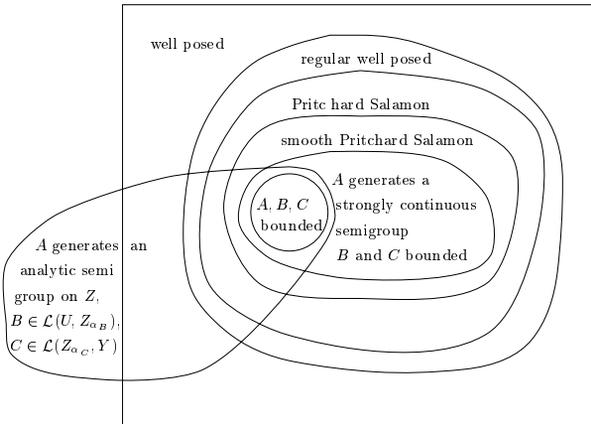


Figure 2.1: Various classes of systems.

## 2.1 Well-posed linear systems

We assume that  $X$  is a Hilbert space and  $A : D(A) \rightarrow X$  is the generator of a strongly continuous semigroup  $\{T(t)\}_{t \geq 0}$  on  $X$ . We define the Hilbert space  $X_1$  as  $D(A)$  with the norm  $\|x\|_1 = \|(\beta I - A)x\|$ , where  $\beta \in \rho(A)$  is fixed<sup>1</sup>. The Hilbert space  $X_{-1}$  is the completion of  $X$  with respect to the norm  $\|x\|_{-1} = \|(\beta I - A)^{-1}x\|$ . We have

$$X_1 \subset X \subset X_{-1}, \quad (2.1)$$

densely and with continuous embeddings. The semigroup  $\{T(t)\}_{t \geq 0}$  extends to a semigroup on  $X_{-1}$ , denoted by the same symbol. The generator of this extended semigroup is an extension of  $A$ , whose domain is  $X$ , so that  $A : X (\subset X_{-1}) \rightarrow X_{-1}$ , see Weiss [86].

We assume that  $U$  is a Hilbert space. The operator  $B \in \mathcal{L}(U, X_{-1})$  is said to be an *admissible control operator* for  $\{T(t)\}_{t \geq 0}$ , if the input maps  $\{\Phi_t\}_{t \geq 0}$  are bounded from  $L_2([0, \infty), U)$  to  $X$  for all finite  $t \geq 0$ , where

$$\Phi_t u := \int_0^t T(t - \tau) B u(\tau) d\tau, \quad \text{for all } u \in L_2([0, \infty), U). \quad (2.2)$$

The above integration is done in  $X_{-1}$ , but the result is in  $X$ . If  $x$  is the solution of  $x(t) = Ax(t) + Bu(t)$ ,  $t \geq 0$ , which is an equation in  $X_{-1}$ , with  $x(0) = x_0 \in X$  and  $u \in L_2([0, \infty), U)$ , then  $x(t) \in X$  for all  $t \geq 0$ . In this case,  $x$  is a continuous  $X$ -valued function of  $t$ . We have that for all  $t \geq 0$ ,

$$x(t) = T(t)x_0 + \Phi_t u. \quad (2.3)$$

The operator  $B$  is called *bounded* if  $B \in \mathcal{L}(U, X)$  (and *unbounded* otherwise).

We assume that  $Y$  is another Hilbert space. The operator  $C \in \mathcal{L}(X_1, Y)$  is called an *admissible observation operator* for  $\{T(t)\}_{t \geq 0}$ , if for every  $t > 0$  there exists a number  $K_t \geq 0$  such that

$$\int_0^t \|CT(\tau)x_0\|^2 d\tau \leq K_t^2 \|x_0\|^2 \quad \text{for all } x_0 \in D(A). \quad (2.4)$$

The operator  $C$  is called *bounded* if it can be extended such that  $C \in \mathcal{L}(X, Y)$ . We regard  $L_2^{1\circ}([0, \infty), Y)$  as a Fréchet space with the seminorms being the  $L_2$ -norms on the intervals  $[0, \mathbf{n}]$ ,  $\mathbf{n} \in \mathbb{N}$ . Then the admissibility of  $C$  means that there is a continuous operator  $\Psi : X \rightarrow L_2^{1\circ}([0, \infty), Y)$  such that

$$(\Psi x_0)(t) = CT(t)x_0 \quad \text{for all } x_0 \in D(A). \quad (2.5)$$

The operator  $\Psi$  is completely determined by (2.5), because  $D(A)$  is dense in  $X$ .

<sup>1</sup>This norm is equivalent to the graph norm.

$B$  is said to be an *infinite-time admissible control operator* for  $\{T(t)\}_{t \geq 0}$  if  $B$  is admissible and for any  $u \in L_2([0, \infty), U)$ , the map  $t \mapsto \Phi_t u$  (from  $[0, \infty)$  to  $X$ ) is bounded.  $C$  is said to be an *infinite-time admissible observation operator* for  $\{T(t)\}_{t \geq 0}$  if there exists a  $K \geq 0$  such that

$$\int_0^\infty \|CT(\tau)x_0\|^2 d\tau \leq K\|x_0\|^2 \text{ for all } x_0 \in D(A).$$

If  $B \in \mathcal{L}(U, X_{-1})$  is an admissible control operator for  $\{T(t)\}_{t \geq 0}$  and  $C \in \mathcal{L}(X_1, Y)$  is an admissible observation operator for  $\{T(t)\}_{t \geq 0}$ , then the *transfer functions* of the system  $\Sigma$  given by the triple  $(A, B, C)$  are solutions  $G : \rho(A) \rightarrow \mathcal{L}(U, Y)$  of

$$G(s) - G(\beta) = -(s - \beta)C(sI - A)^{-1}(\beta I - A)^{-1}B \tag{2.6}$$

for  $s$  and  $\beta$  in  $\rho(A)$ . We remark that since  $B$  is an admissible control operator for  $\{T(t)\}_{t \geq 0}$ ,  $(\cdot I - A)^{-1}B$  is an  $\mathcal{L}(U, X)$ -valued analytic function and since  $C$  is an admissible observation operator  $\{T(t)\}_{t \geq 0}$ ,  $C(\cdot I - A)^{-1}$  is a  $\mathcal{L}(X, Y)$ -valued analytic function. Both  $(\cdot I - A)^{-1}B$  and  $C(\cdot I - A)^{-1}$  are analytic on some right half-plane  $\mathbb{C}_\alpha^+ = \{s \in \mathbb{C} \mid \operatorname{Re}(s) > \alpha\}$ . Thus any transfer function is a  $\mathcal{L}(U, Y)$ -valued function which is analytic in some  $\mathbb{C}_\alpha^+$ . Moreover any two transfer functions differ only by an additive constant,  $D \in \mathcal{L}(U, Y)$ . The point is that they need not necessarily be bounded on any  $\mathbb{C}_\alpha^+$ . We impose this as an extra assumption on the triple  $(A, B, C)$  and call this well-posedness. The system  $\Sigma$  given by the triple  $(A, B, C)$  is said to be *well-posed linear system* if  $B$  is an admissible control operator for  $\{T(t)\}_{t \geq 0}$ ,  $C$  is an admissible observation operator for  $\{T(t)\}_{t \geq 0}$  and its transfer functions are bounded on some half-plane  $\mathbb{C}_\alpha^+$ . For a well-posed linear system, the operator from the initial state and the input function to the final state and the output function is bounded. The input and output functions  $u$  and  $y$  are locally  $L_2$  functions with values in  $U$  and in  $Y$ , respectively. The state trajectory  $x$  is an  $X$ -valued function. The boundedness property mentioned earlier means that for every  $t > 0$  there is a  $c_t \geq 0$  such that

$$\|x(t)\|^2 + \int_0^t \|y(\tau)\|^2 d\tau \leq c_t^2 \left[ \|x(0)\|^2 + \int_0^t \|u(\tau)\|^2 d\tau \right] \tag{2.7}$$

(with  $c_t$  independent of  $x(0)$  and of  $u$ ). For the detailed definition, background and examples we refer to Salamon [72], [73], Staffans [75], [76], [79], Weiss [89], [88], Weiss and Rebarber [68] and Weiss<sup>2</sup> [90].

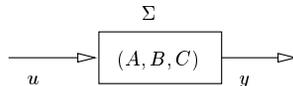


Figure 2.2: Well-posed linear systems.

$\{T(t)\}_{t \geq 0}$  is called the *semigroup* of  $\Sigma$  and  $A$  is called its *infinitesimal generator*.  $U$  is called the *input space*,  $Y$  is called the *output space* and  $X$  is called the

*state-space* of the well-posed linear system. The well-posed linear system  $\Sigma$  is called *regular* if the limit

$$\lim_{\lambda \rightarrow +\infty} G(\lambda)v = Dv \quad (2.8)$$

exists for every  $v \in U$ , where  $\lambda$  is real (see [87], [89]). In this case, the operator  $D \in \mathcal{L}(U, Y)$  is called the *feedthrough operator* of  $\Sigma$ , and there holds

$$G(s) = C_\Lambda(sI - A)^{-1}B + D,$$

where  $C_\Lambda$  is an extension of  $C$  given by

$$C_\Lambda x_0 = \lim_{\lambda \rightarrow +\infty} C\lambda(\lambda I - A)^{-1}x_0 \quad (2.9)$$

( $\lambda \in \mathbb{R}$ ), for all  $x_0$  in the domain

$$D(C_\Lambda) = \{x_0 \in X \mid \text{the limit in (2.9) exists}\}$$

(see Staffans [79]). The operators  $A, B, C, D$  are called the *generating operators* of  $\Sigma$ , and we write  $\Sigma = (A, B, C, D)$ .

## 2.2 Duality theory

We will use the notion of “duality” in this book following the notation and results from Keulen [84]. For a short introduction about representations of dual spaces, duality pairings and pivot spaces we refer to Aubin [4].

The dual space of a Hilbert space  $Z$  is just the linear space of bounded linear functionals, that is,  $\mathcal{L}(Z, \mathbb{C})$ . We shall denote  $\mathcal{L}(Z, \mathbb{C})$  by  $Z^d$ . The vector space  $Z^d$  is a Banach space with the dual norm  $\|\cdot\|_d$  defined as follows: If  $z' \in Z^d$ , then

$$\|z'\|_d = \sup_{z \in Z, z \neq 0} \frac{|z'(z)|}{\|z\|}.$$

If  $Z$  is a Hilbert space, then there exists a surjective isometry  $J$  from  $Z$  to  $Z^d$ , defined by  $(J(z_1))(z_2) = \langle z_2, z_1 \rangle_Z$  for all  $z_2 \in Z$ . The dual space  $Z^d$  is a Hilbert space with the inner product defined as follows: if  $z'_1$  and  $z'_2 \in Z^d$ , then

$$\langle z'_1, z'_2 \rangle_{Z^d} = \langle J^{-1}z'_2, J^{-1}z'_1 \rangle_Z = z'_1(J^{-1}(z'_2)).$$

We call the isometry  $J : Z \rightarrow Z^d$  the *duality map*<sup>2</sup> from  $Z$  to its dual  $Z^d$ . The inverse map  $J^{-1} : Z^d \rightarrow Z$  is denoted by  $\iota_Z$  and the inner product on  $Z^d$  satisfies

$$\langle z'_1, z'_2 \rangle_{Z^d} = \langle \iota_Z z'_2, \iota_Z z'_1 \rangle_Z \quad \text{for all } z'_1 \text{ and } z'_2 \text{ in } Z^d.$$

---

<sup>2</sup>The duality map is *conjugate linear*, that is, it satisfies  $J(\alpha_1 z_1 + \alpha_2 z_2) = \overline{\alpha_1} J(z_1) + \overline{\alpha_2} J(z_2)$  for all  $\alpha_1$  and  $\alpha_2$  in  $\mathbb{C}$ , and all  $z_1$  and  $z_2$  in  $Z$ .

The duality pairing  $\langle \cdot, \cdot \rangle_{\langle Z^d, Z \rangle}$  is defined by

$$\langle z', z \rangle_{\langle Z^d, Z \rangle} = z'(z) = \langle z, \iota_Z z' \rangle_Z, \text{ for all } z' \in Z^d \text{ and } z \in Z.$$

Let  $Z'$  be a Hilbert space and let  $j$  be a conjugate linear map which is an isometry from  $Z'$  onto  $Z^d$ , that is,

$$Z' \xrightarrow{j} Z^d \xrightarrow{\iota_Z} Z$$

and  $\iota_Z j \in \mathcal{L}(Z', Z)$  is an isometry onto  $Z$ . The pair  $\{Z', j\}$  is called a *representation of  $Z^d$* . We define the *duality pairing*  $\langle \cdot, \cdot \rangle_{\langle Z', Z \rangle}$  by

$$\langle z', z \rangle_{\langle Z', Z \rangle} = \langle jz', z \rangle_{\langle Z^d, Z \rangle} = (jz')(z) = \langle z, \iota_Z jz' \rangle_Z,$$

for all  $z' \in Z'$  and  $z \in Z$ . In this case, we also call  $Z'$  the dual space of  $Z$  (this is standard abuse of terminology).

We say that  $Z$  is identified with its dual if the representation for  $Z^d$  is chosen to be  $\{Z, \iota_Z^{-1}\}$ . It follows that in this case, the duality pairing corresponds to the inner product on  $Z$ :

$$\langle z_1, z_2 \rangle_{\langle Z, Z \rangle} = \langle z_2, \iota_Z \iota_Z^{-1} z_1 \rangle_Z = \langle z_2, z_1 \rangle_Z$$

for all  $z_1$  and  $z_2$  in  $Z$ . If  $Z$  is identified with its dual,  $Z$  is called a *pivot space*.

Usually, it is assumed that whatever choice is made for the representation  $\{Z', j\}$  of  $\mathcal{L}(Z, \mathbb{C})$ , the representation of the dual of this dual space (that is,  $(Z')^d = \mathcal{L}(Z', \mathbb{C})$ ) is given by  $\{Z, c_Z\}$ , where  $c_Z$  is the isometry from  $Z$  to  $Z^d$  given by  $c_Z = \iota_Z^{-1} j^{-1} \iota_Z^{-1}$ :

$$(Z')^d \xrightarrow{\iota_{Z'}} Z' \xrightarrow{j} Z^d \xrightarrow{\iota_Z} Z$$

(we could also say that  $Z'' = Z$ ). This implies that

$$\begin{aligned} \langle z'', z' \rangle_{\langle Z'', Z' \rangle} &= \langle z'', z' \rangle_{\langle Z, Z' \rangle} \\ &= \langle c_Z z'', z' \rangle_{\langle (Z')^d, Z' \rangle} \\ &= (c_Z z'')(z') \\ &= (\iota_Z^{-1} j^{-1} \iota_Z^{-1} z'')(z') \\ &= \langle z', j^{-1} \iota_Z^{-1} z'' \rangle_{Z'} \\ &= \langle \iota_Z^{-1} z'', jz' \rangle_{Z^d} \\ &= \langle \iota_Z jz', z'' \rangle_Z \\ &= \overline{(jz')(z'')} \\ &= \overline{\langle z', z'' \rangle_{\langle Z', Z \rangle}}. \end{aligned}$$

We identify the bidual of  $Z$  with  $Z$  itself and the duality pairing  $\langle \cdot, \cdot \rangle_{\langle Z, Z' \rangle}$  is given by

$$\langle z'', z' \rangle_{\langle Z, Z' \rangle} = \overline{\langle z', z'' \rangle_{\langle Z', Z \rangle}}.$$

If we have two Hilbert spaces  $W$  and  $Z$  with  ${}^3 W \hookrightarrow Z$ , then  $Z^d \hookrightarrow W^d$ . Here, we identify an element  $z' \in Z^d = \mathcal{L}(Z, \mathbb{C})$  with  $z'|_W \in W^d = \mathcal{L}(W, \mathbb{C})$ . Similarly, if  $w' \in W^d$  satisfies  $|w'(z)| \leq (\text{a constant}) \|z\|_Z$  for all  $z \in W$ , then the unique continuous extension of  $w'$  to  $Z$  is also denoted by  $w'$ . Furthermore, for all  $z' \in Z^d$  and  $w \in W$  we have

$$\langle z', w \rangle_{\langle Z^d, Z \rangle} = z'(w) = \langle z', w \rangle_{\langle W^d, W \rangle}.$$

Now if  $Z$  is identified with its dual, we can define a space  $W'$  which is a representation of the dual of  $W$  such that  $W \hookrightarrow Z = Z' \hookrightarrow W'$ , as follows. Recall that  $\iota_Z$  denotes the canonical map from  $Z^d$  to  $Z$ . Let  $W'$  be the completion of  $Z' = Z$  with respect to the inner product

$$\langle w'_1, w'_2 \rangle_{W'} = \langle \iota_Z^{-1} w'_2, \iota_Z^{-1} w'_1 \rangle_{Z^d}$$

for  $w'_1$  and  $w'_2$  in  $Z = Z'$ . We have

$$\begin{array}{ccc} Z^d & \hookrightarrow & W^d \\ \iota_Z \downarrow & & \\ Z = Z' & \hookrightarrow & W'. \end{array}$$

For all  $z' \in Z^d$ ,

$$\|\iota_Z z'\|_{W'} = \|\iota_Z^{-1} \iota_Z z'\|_{Z^d} = \|z'\|_{W^d}.$$

Thus it follows that  $\iota_Z$  has a unique conjugate linear extension  $\overline{\iota_Z}$  to  $W^d$  and  $\overline{\iota_Z}$  is an isometry from  $W^d$  to  $W'$ . Furthermore, we can choose the pair  $\{W', (\overline{\iota_Z})^{-1}\}$  as a representation of  $W^d$  so that the duality pairing  $\langle \cdot, \cdot \rangle_{\langle W', W \rangle}$  is given by

$$\langle w', w \rangle_{\langle W', W \rangle} = (\overline{\iota_Z}^{-1} w') (w)$$

and for all  $(z', w) \in Z \times W$  we have

$$\langle z', w \rangle_{\langle W', W \rangle} = (\overline{\iota_Z}^{-1} z') (w) = (\iota_Z^{-1} z') (w) = \langle w, z' \rangle_Z.$$

Finally,  $W \hookrightarrow Z = Z' \hookrightarrow W'$ .

If in addition we have a third Hilbert space  $V$  with  $Z \hookrightarrow V$ , we have  $V^d \hookrightarrow Z^d$  and we can define a representation  $V'$  of  $V^d$  such that  $V' \hookrightarrow Z = Z' \hookrightarrow V$  as follows. Let  $\iota_Z|_{V^d}$  denote the restriction of  $\iota_Z$  to  $V^d$  and define  $V'$  as the image of  $V^d$  under this map. Furthermore, define the inner product on  $V'$  by

$$\langle v'_1, v'_2 \rangle_{V'} = \left\langle (\iota_Z|_{V^d})^{-1} v'_2, (\iota_Z|_{V^d})^{-1} v'_1 \right\rangle_{Z^d}.$$

Then the pair  $\{V', (\iota_Z|_{V^d})^{-1}\}$  is a representation of  $V^d$  and the duality pairing is given by

$$\langle v', v \rangle_{\langle V', V \rangle} = \left( (\iota_Z|_{V^d})^{-1} v' \right) (v).$$

---

<sup>3</sup>Here we use the notation  $\hookrightarrow$  to mean  $W \subset Z$ ,  $W$  is dense in  $Z$ , and the canonical injection  $x \mapsto x : W \rightarrow Z$  is continuous.

Hence for all  $(v', z) \in V' \times Z$  we have

$$\langle v', z \rangle_{\langle V', V \rangle} = \left( (\iota_Z|_{V^d})^{-1} v' \right) (z) = (\iota_Z^{-1} v') (z) = \langle z, v' \rangle_Z$$

and  $V' \hookrightarrow Z' = Z \hookrightarrow V$ . We depict this schematically in Figure 2.3. For all  $(v', w) \in V' \times W$  we have

$$\langle v', w \rangle_{\langle W', W \rangle} = \langle w, v' \rangle_Z = \langle v', w \rangle_{\langle V', V \rangle}.$$

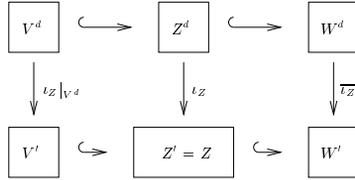


Figure 2.3: Interlacing of the dual spaces.

Suppose that  $X$  and  $Y$  are Hilbert spaces and that  $\{X', j\}$  and  $\{Y', k\}$  are representations of  $X^d$  and  $Y^d$ , with the pairings denoted by  $\langle \cdot, \cdot \rangle_{\langle X', X \rangle}$  and  $\langle \cdot, \cdot \rangle_{\langle Y', Y \rangle}$ , respectively. Note that now  $\iota_X j$  is an isometry from  $X'$  to  $X$  and that  $\iota_Y k$  is an isometry from  $Y'$  to  $Y$ . The dual of a densely defined operator  $A$  from  $D(A) \subset X$  to  $Y$  is an operator  $A'$  from  $D(A') \subset Y'$  to  $X'$ , defined as follows:

1.  $D(A') = \left\{ y' \in Y' \mid \exists x' \in X' \text{ such that for all } x \in D(A) \right. \\ \left. \langle y', Ax \rangle_{\langle Y', Y \rangle} = \langle x', x \rangle_{\langle X', X \rangle} \right\}$ , and
2.  $\langle A' y', x \rangle_{\langle X', X \rangle} = \langle y', Ax \rangle_{\langle Y', Y \rangle}$  for all  $y' \in D(A')$  and all  $x \in D(A)$ .

The adjoint of  $A$  (denoted by  $A^*$ ) from  $D(A^*) \subset Y$  to  $X$  is related to  $A'$  by

$$\begin{aligned} D(A^*) &= (\iota_Y k) D(A'), \\ A^* y &= (\iota_X j) A' (\iota_Y k)^{-1} y \text{ for all } y \in D(A^*), \end{aligned}$$

so that for all  $y \in D(A^*)$  and  $x \in D(A)$  we have

$$\begin{aligned} \langle x, A^* y \rangle_X &= \langle (\iota_X j)^{-1} (\iota_X j) A' (\iota_Y k)^{-1} y, x \rangle_{\langle X', X \rangle} \\ &= \langle (\iota_Y k)^{-1} y, Ax \rangle_{\langle Y', Y \rangle} \\ &= \langle Ax, y \rangle_Y. \end{aligned}$$

For a bounded linear operator  $T \in \mathcal{L}(X, Y)$  we have  $T' \in \mathcal{L}(Y', X')$ ,  $T^* = (\iota_X j) T' (\iota_Y k)^{-1} \in \mathcal{L}(Y, X)$ . Owing to the identification of the bidual of a Hilbert space with itself, we have that an operator  $T \in \mathcal{L}(X, Y)$  satisfies  $(T')' = T$ .

The dual of a linear densely defined closed operator  $A : D(A) (\subset X) \rightarrow Y$ , denoted by  $A' : D(A') (\subset Y') \rightarrow X'$  is closed and densely defined. Hence the

dual of  $A'$ , denoted by  $(A')' : D((A')') (\subset X'' = X) \rightarrow Y'' = Y$  is well-defined and moreover,  $D((A')') = D(A)$  and  $(A')'x = Ax$  for all  $x \in D(A)$ .

Since we identify the bidual of a Hilbert space with itself, it follows that the dual of an operator  $T \in \mathcal{L}(X, X')$  satisfies  $T' \in \mathcal{L}(X, X')$ . We call an operator  $T \in \mathcal{L}(X, X')$  self-dual if  $T = T'$ . In this case the operator  $S := (\iota_{X'}j)T \in \mathcal{L}(X)$  is self-adjoint and satisfies  $\langle y, Sx \rangle_X = \langle Tx, y \rangle_{\langle X', X \rangle}$ .

## 2.3 The Pritchard-Salamon class

In this section we introduce a special class of well-posed linear systems for which there exists a rich literature (see Salamon [72], [73], Pritchard and Salamon [66], Curtain et al. [15]). Most systems containing delays and some systems described by partial differential equations fit into this framework. This class generalizes the class of state linear systems with bounded input and output operators studied in [27].

Let  $V$  and  $W$  be separable Hilbert spaces with continuous, dense injections and which satisfy

$$W \hookrightarrow Z \hookrightarrow V.$$

Suppose that  $A$  is the infinitesimal generator of strongly continuous semigroups  $\{T^W(t)\}_{t \geq 0}$ ,  $\{T^Z(t)\}_{t \geq 0}$  and  $\{T^V(t)\}_{t \geq 0}$  on  $W$ ,  $Z$  and  $V$ , respectively, such that

$$T^V(t)|_Z = T^Z(t) \text{ and } T^Z(t)|_W = T^W(t).$$

Since these semigroups are consistent, we shall simply use the notation  $\{T(t)\}_{t \geq 0}$ . Assume further that  $U$  and  $Y$  are separable Hilbert spaces (the input and output spaces), respectively.

1.  $B \in \mathcal{L}(U, V)$  is a *Pritchard-Salamon admissible control operator* for  $\{T(t)\}_{t \geq 0}$  if there exist a  $t > 0$  and a constant  $\beta > 0$  such that

$$\left\| \int_0^t T(t-\tau)Bu(\tau)d\tau \right\|_W \leq \beta \|u\|_{L_2([0,t],U)} \quad (2.10)$$

for all  $u \in L_2([0, t], U)$ .

2.  $C \in \mathcal{L}(W, Y)$  is a *Pritchard-Salamon admissible observation operator* for  $\{T(t)\}_{t \geq 0}$  if there exist a  $t > 0$  and a constant  $\gamma > 0$  such that

$$\|CT(\cdot)z\|_{L_2([0,t],Y)} \leq \gamma \|z\|_V \quad (2.11)$$

for all  $z \in W$ .

**Remarks:** In the above definition,  $Z$  is not essential and in the applications it is usual to take either  $Z = V$  or  $Z = W$ . Furthermore, we remark that if (2.10) holds for some  $t$ , then it can be shown that it holds for all  $t > 0$ , where  $\beta$  will depend on  $t$ . Similarly, if (2.11) holds for some  $t$ , then it can be shown that it holds for all  $t > 0$ , where  $\gamma$  will depend on  $t$  in general. See for example, Remark 2.10, page 11, Curtain et al. [15].

Under the above assumptions, the state linear system  $\Sigma(A, B, C, D)$  is called a *Pritchard-Salamon system* for any  $D \in \mathcal{L}(U, Y)$ . If, in addition,  $D(A^V) \hookrightarrow W$ , then  $\Sigma(A, B, C, D)$  is called a *smooth Pritchard-Salamon system*.

When confusion may arise we use superscripts, for example  $A^X$ ,  $T^X(t)$ , to denote the operators on  $X = V$ ,  $Z$  or  $W$ . We denote the growth bound<sup>4</sup> on  $X$  by  $\omega_0^X$  (the growth bounds  $\omega_0^W$  and  $\omega_0^V$  are different in general). We now enumerate a few properties of Pritchard-Salamon systems which have been proved in Curtain et al. [15] and in Curtain and Zwart [26] and will be used in the proofs.

1. For a smooth Pritchard-Salamon system,  $\sigma_p(A^W) = \sigma_p(A^V)$  and  $\sigma(A^W) = \sigma(A^V)$ .
2. Let  $L \in \mathcal{L}(Y, V)$  be a Pritchard-Salamon admissible input operator for  $\{T(t)\}_{t \geq 0}$  and assume that  $\Sigma(A, B, C, D)$  is a Pritchard-Salamon system. Then  $\Sigma(A + LC, B, C, D)$  is also a Pritchard-Salamon system, where  $A + LC$  is the infinitesimal generator of  $\{T_{LC}(t)\}_{t \geq 0}$  which is the unique solution of

$$T_{LC}(t)z = T(t)z + \int_0^t T_{LC}(t - \tau)LCT(\tau)z d\tau.$$

Furthermore,  $\{T_{LC}(t)\}_{t \geq 0}$  is also the unique solution of

$$T_{LC}(t)z = T(t)z + \int_0^t T(t - \tau)LCT_{LC}(\tau)z d\tau.$$

We note that  $L \in \mathcal{L}(Y, W)$  is always a Pritchard-Salamon admissible control operator.

3. A smooth Pritchard-Salamon system  $\Sigma(A, B, C, D)$  has a well-defined transfer operator  $G(s)$  given by  $\hat{y}(s) = G(s)\hat{u}(s)$ , for all  $s \in \mathbb{C}_\alpha^+ := \{s \in \mathbb{C} \mid \operatorname{Re}(s) > \alpha\}$ , where  $\alpha > \max\{\omega_0^V, \omega_0^W\}$ , and

$$G(s) = D + C(sI - A^V)^{-1}B.$$

---

<sup>4</sup>The *growth bound* of a semigroup  $\{T(t)\}_{t \geq 0}$ , denoted by  $\omega_0$ , is defined to be  $\inf_{t > 0} \frac{\log \|T(t)\|}{t}$

Furthermore,  $C(\cdot I - A^V)^{-1} B \in H_\infty(\mathbb{C}_\alpha^+, \mathcal{L}(U, Y))$ , where

$$H_\infty(\mathbb{C}_\alpha^+, \mathcal{L}(U, Y)) = \left\{ G : \mathbb{C}_\alpha^+ \rightarrow \mathcal{L}(U, Y) \mid \begin{array}{l} G \text{ is analytic, and} \\ \sup_{s \in \mathbb{C}_\alpha^+} \|G(s)\|_{\mathcal{L}(U, Y)} < \infty \end{array} \right\}.$$

4. Let us assume that the Hilbert spaces  $Z, U, Y$  are identified with their duals. Then  $B' \in \mathcal{L}(V', U)$ ,  $C' \in \mathcal{L}(Y, W')$ ,  $D' \in \mathcal{L}(Y, U)$  and  $V' \hookrightarrow Z' \hookrightarrow W'$ .

If  $\Sigma(A, B, C, D)$  is a smooth Pritchard-Salamon system, then  $\Sigma(A', C', B', D')$  is also a smooth Pritchard-Salamon system, and  $D \left( (A')^{W'} \right) \hookrightarrow V'$ .

If  $A$  generates an exponentially stable strongly continuous semigroup on  $V$  and  $W$ , then we say that the Pritchard-Salamon system  $\Sigma(A, B, C, D)$  is *exponentially stable*. Its *controllability map*  $\mathcal{B} \in \mathcal{L}(L_2([0, \infty), U), W)$  is defined by

$$\mathcal{B}u = \int_0^\infty T(s)Bu(s)ds$$

and its *observability map*  $\mathcal{C} \in \mathcal{L}(V, L_2([0, \infty), Y))$  is defined by

$$\mathcal{C}z = CT(\cdot)z$$

for  $z \in W$ . The *controllability Gramian*  $L_B \in \mathcal{L}(W', W)$  and *observability Gramian*  $L_C \in \mathcal{L}(V, V')$  are defined by  $L_B = \mathcal{B}\mathcal{B}'$ ,  $L_C = \mathcal{C}'\mathcal{C}$ . Its *Hankel operator*  $\Gamma \in \mathcal{L}(L_2([0, \infty), U), L_2([0, \infty), Y))$  is defined by  $\Gamma = \mathcal{C}\mathcal{B}$ . For an exponentially stable, smooth Pritchard-Salamon system,  $C(sI - A)^{-1}B \in H_\infty^c(\mathcal{L}(U, Y))$ , where  $H_\infty^c(\mathcal{L}(U, Y))$  denotes the space of maps  $G$  defined in the closed right half-plane  $\text{Re}(s) \geq 0$ , taking values in  $\mathcal{L}(U, Y)$ , such that  $G$  is analytic in the open right half-plane  $\text{Re}(s) > 0$ , and  $G$  is bounded and continuous in the closed right half-plane  $\text{Re}(s) \geq 0$ . Moreover,

$$\lim_{\substack{|s| \rightarrow \infty \\ \text{Re}(s) \geq 0}} \|G(s)\| = 0. \quad (2.12)$$

(That  $e^{\epsilon t}CT(\cdot)B \in L_2([0, \infty), \mathcal{L}(U, Y))$  for some  $\epsilon > 0$  is shown in Curtain et al. [15], where  $CT(\cdot)B$  has to be interpreted in the sense of Remark 2.10.(iv) [15]. Using the Cauchy-Schwarz inequality, it follows that  $CT(\cdot)B \in L_1([0, \infty), \mathcal{L}(U, Y))$ . Finally, using Property A.6.2.g, page 636, Curtain and Zwart [27], we obtain the desired result.)

We now quote the following lemma from Curtain and Zwart [26] on Lyapunov equations:

**Lemma 2.3.1** *Suppose that  $\Sigma(A, B, C, D)$  is an exponentially stable, smooth Pritchard-Salamon system. Then*

1.  $L_B$  and  $L_C$  are the unique solutions in  $\mathcal{L}(W', W)$  and  $\mathcal{L}(V, V')$  respectively of the following Lyapunov equations

$$\langle z_1, (AL_B + L_B A' + BB')z_2 \rangle_{\langle W', W \rangle} = 0 \quad (2.13)$$

for all  $z_1, z_2 \in D\left((A')^{W'}\right)$ ;

$$\langle z_1, (A' L_C + L_C A + C' C)z_2 \rangle_{\langle V, V' \rangle} = 0 \quad (2.14)$$

for all  $z_1, z_2 \in D(A^V)$ . Moreover,

$$L_B \in \mathcal{L}\left(D\left((A')^{W'}\right), D(A^V)\right)$$

and

$$L_C \in \mathcal{L}\left(D(A^V), D\left((A')^{W'}\right)\right).$$

2.  $L_B L_C \in \mathcal{L}(V, W) \cap \mathcal{L}(V) \cap \mathcal{L}(W)$ ,  $L_C L_B \in \mathcal{L}(W', V') \cap \mathcal{L}(V') \cap \mathcal{L}(W')$ , and  $\rho^V(L_B L_C) = \rho^W(L_B L_C)$  and for  $-\frac{1}{\alpha} \in \rho(L_B L_C)$ , there holds

$$\begin{aligned} (I + \alpha L_B L_C)^{-1} &\in \mathcal{L}(V) \cap \mathcal{L}(W), \\ (I + \alpha L_C L_B)^{-1} &\in \mathcal{L}(W') \cap \mathcal{L}(V') \end{aligned}$$

and

$$\begin{aligned} (I + \alpha L_B L_C)^{-1} &\in \mathcal{L}(D(A^V)), \\ (I + \alpha L_C L_B)^{-1} &\in \mathcal{L}\left(D\left((A')^{W'}\right)\right). \end{aligned}$$

From Curtain and Zwart [26], we have

**Lemma 2.3.2** *Suppose that  $\Sigma(A, B, C)$  is an exponentially stable, smooth Pritchard-Salamon system. Then  $\sigma(\Gamma^* \Gamma) \setminus \{0\} = \sigma(L_B L_C) \setminus \{0\}$  and  $r(L_C) = r(\Gamma^* \Gamma) = \|\Gamma\|^2$ , where  $r(\cdot)$  denotes the spectral radius.*

If  $U = \mathbb{C}^m$  and  $Y = \mathbb{C}^p$ , then the Hankel operator of an exponentially stable Pritchard-Salamon system is compact (Lemma 3.5 in Curtain et al. [15] and Glover et al. [37]).

**Lemma 2.3.3** *Suppose that  $\Sigma(A, B, C)$  is an exponentially stable, smooth Pritchard-Salamon system with input space  $\mathbb{C}^m$  and output space  $\mathbb{C}^p$  and let  $\sigma_{1+1} < \sigma < \sigma_1$ . The operator  $J := I - \frac{1}{\sigma^2} \Gamma^* \Gamma \in \mathcal{L}(L_2([0, \infty), \mathbb{C}^m))$  has a spectrum  $\sigma(J)$  contained in  $(-\infty, -\delta) \cup (\delta, \infty)$  for some  $\delta > 0$ , and  $\sigma(J) \cap (-\infty, -\delta)$  consists of exactly 1 negative eigenvalues.*

**Proof**  $\Gamma^*\Gamma$  is compact and has a pure point spectrum with 0 as the accumulation point. Considering the resolvent  $[(I - \frac{1}{\sigma^2}\Gamma^*\Gamma) - \lambda I]^{-1}$ , it is easy to see that  $J$  has a spectrum which is a shifted version of the spectrum of  $\frac{1}{\sigma^2}\Gamma^*\Gamma$ . Finally, since  $\Gamma^*\Gamma$  has a pure point spectrum  $\{\sigma_1^2, \sigma_2^2, \dots\}$  with 0 as the accumulation point, and since  $\sigma_{1+1} < \sigma < \sigma_1$ ,  $J$  has exactly 1 negative eigenvalues. ■

Finally we give a simple example<sup>5</sup> of a delay system which can be modelled as a smooth Pritchard-Salamon system.

**Example 2.3.4** Let  $G(s) = \frac{e^{-s\tau}}{s+1}$ , where  $\tau > 0$ . We will first give a realization of  $G$ . Let  $Z = \mathbb{C} \times L_2(-\tau, 0)$  (with the obvious inner product). Consider the Sobolev space

$$W^{1,2}(-\tau, 0) = \left\{ f \in L_2(-\tau, 0) \mid \begin{array}{l} \text{the derivative of } f \text{ (in the sense of} \\ \text{distributions) is a regular distribution} \\ T_g, \text{ with } g \in L_2(-\tau, 0) \end{array} \right\},$$

equipped with the inner product

$$\langle f, g \rangle_{W^{1,2}(-\tau, 0)} = \int_{-\tau}^0 [f(x)\overline{g(x)} + f'(x)\overline{g'(x)}] dx.$$

(With this inner product,  $W^{1,2}(-\tau, 0)$  is a Hilbert space: see for example, Proposition 5, page 55, Yosida [91]. Furthermore, from Corollary 7.3 [80], it follows that  $W^{1,2}(-\tau, 0) \subset \mathcal{C}(-\tau, 0)$ .) Define

$$A \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -z_2(0) \\ z_2' \end{bmatrix},$$

with

$$D(A) = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in Z \mid z_2 \in W^{1,2}(-\tau, 0), z_1 = z_2(0) \right\}.$$

Then it can be shown that  $A$  is the infinitesimal generator of an exponentially stable strongly continuous semigroup  $\{T(t)\}_{t \geq 0}$  on  $Z$  (see for instance, Example 2.8, van Keulen [84]). The space  $W := D(A)$  with the inner product

$$\left\langle \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix} \right\rangle_W = \langle z_2, \tilde{z}_2 \rangle_{W^{1,2}(-\tau, 0)}$$

is a Hilbert space and  $\{T(t)\}_{t \geq 0}$  restricts to a strongly continuous semigroup on  $W$ . Let  $B \in \mathcal{L}(\mathbb{C}, Z)$  be defined as follows:

$$Bu = \begin{bmatrix} u \\ 0 \end{bmatrix},$$

---

<sup>5</sup>  $\begin{cases} z(t) = -z(t) + u(t), \\ y(t) = z(t - \tau). \end{cases}$

Finally, if  $C : D(A) \rightarrow \mathbb{C}$  is given by

$$C \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = z_2(-\tau),$$

then  $C \in \mathcal{L}(W, \mathbb{C})$ . We now choose  $V = Z$ . In Pritchard and Salamon [66] it is explained that  $B$  and  $C$  are Pritchard-Salamon admissible control and observation operators, respectively. Finally we note that  $D(A^V) \hookrightarrow W$  is trivially satisfied since  $W = D(A)$ , and so in fact we have a smooth Pritchard-Salamon system, with a transfer function  $\frac{e^{-\tau s}}{s+1}$ .  $\diamond$

## 2.4 The analytic class

In this section we introduce the class of exponentially stable analytic infinite-dimensional systems and establish some basic system theoretic properties.

First we recall the basic properties of exponentially stable analytic semigroups using the notation from Staffans [77] (see also Pazy [63]).

1. Let  $-A$  be the infinitesimal generator of an exponentially stable analytic semigroup  $\{T(t)\}_{t \geq 0}$  in  $Z$ .
2. For each  $\theta \in \mathbb{R}$ , we let<sup>6</sup>  $Z_\theta = A^{-\theta}Z$  be  $D(A^\theta)$  with the norm  $\|z\|_{Z_\theta} = \|A^\theta z\|_Z$  and inner product  $\langle z_1, z_2 \rangle_{Z_\theta} = \langle A^\theta z_1, A^\theta z_2 \rangle_Z$ .
3. The restrictions of  $-A$  to  $Z_\theta$  for  $\theta > 0$  and the extensions of  $-A$  to  $Z_\theta$  for  $\theta < 0$  (which we still denote by  $-A$ ) generate analytic semigroups in  $Z_\theta$  for all  $\theta \in \mathbb{R}$ . The generator of the semigroup  $\{T(t)\}_{t \geq 0}$  on  $Z_\theta$  is then  $-A \in \mathcal{L}(Z_{\theta+1}, Z_\theta)$ .

These semigroups are all similar to each other, and they commute with  $A^\beta$  for all  $\beta \in \mathbb{R}$ . We therefore denote all of them by the same symbol  $\{T(t)\}_{t \geq 0}$ .

4. For each  $t > 0$  and  $\theta \in \mathbb{R}$ ,  $T(t)$  maps  $Z_\theta$  into  $\bigcap_{\beta \in \mathbb{R}} Z_\beta$ .
5. For each  $\theta \geq 0$ , there exist  $K_1 > 0$  and  $\epsilon > 0$  such that

$$\|A^\theta T(t)\| \leq K_1 \frac{e^{-\epsilon t}}{t^\theta}, \quad t > 0$$

where the norm represents the operator norm in any one of the spaces  $Z_\theta$ .

If  $\theta < 0$ , then there exist  $K_2 > 0$  and  $\epsilon > 0$  such that

$$\|A^\theta T(t)\|_{\mathcal{L}(Z)} \leq K_2 e^{-\epsilon t}, \quad t > 0.$$

(This follows from Lemma 6.3, page 71, Pazy [63].)

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<sup>6</sup>For the definition of  $A^\theta$  and its related properties, we refer the reader to §2.6 of Pazy [63].

6. The map  $t \mapsto T(t) : (0, \infty) \rightarrow \mathcal{L}(Z)$  is continuous in the norm topology. (See for example, Proposition 2.1.1. (iv), page 35, Lunardi, [53].)

This implies that for any  $\theta \in \mathbb{R}$ , the map  $t \mapsto T(t) : (0, \infty) \rightarrow \mathcal{L}(Z_\theta)$  is continuous in the norm topology.

7. The same conclusion as above can be repeated with  $A$  replaced by  $A'$  to give another chain of Hilbert spaces  $Z'_\theta = (A')^{-\theta} Z$  with similar properties.

We identify  $Z'_\theta$  with the dual of  $Z_{-\theta}$  by using  $Z$  as the pivot space. Note that  $Z_0 = Z'_0 = Z$ . For  $z_1 \in Z'_{-\theta}$  and  $z_2 \in Z_\theta$ , we have

$$\langle z_1, z_2 \rangle_{\langle Z'_{-\theta}, Z_\theta \rangle} = \left\langle A^\theta z_2, (A')^{-\theta} z_1 \right\rangle_Z,$$

where  $\langle \cdot, \cdot \rangle_{\langle Z'_{-\theta}, Z_\theta \rangle}$  denotes the duality pairing.

Next we analyze the system theoretic properties of the class of systems described by the triple  $(-A, B, C)$  on the Hilbert space  $Z$ , under the assumptions

- A1.  $-A$  is the infinitesimal generator of an exponentially stable analytic semigroup  $\{T(t)\}_{t \geq 0}$  on the Hilbert space  $Z$ .
- A2.  $B \in \mathcal{L}(U, Z_\alpha)$ , where  $U$  is a Hilbert space and  $\alpha$  is a fixed number satisfying  $-1 < \alpha \leq 0$ .
- A3.  $C \in \mathcal{L}(Z, Y)$ , where  $Y$  is a Hilbert space.

We note that the assumption A2 implies that  $A^\alpha B \in \mathcal{L}(U, Z)$  and  $B \in \mathcal{L}(U, Z_\beta)$  for every  $\beta$  satisfying  $-1 < \beta \leq \alpha$ :  $Z \hookrightarrow Z_\alpha \hookrightarrow Z_\beta \hookrightarrow Z_{-1}$ .

**Remark:** We remark that what follows is similar to what was done in Curtain and Ichikawa [14], but there are differences in assumptions (see Table 2.1).

First we derive several results about  $B$  and its controllability Gramian. In Lemma 2.4.1.1 below we prove that if  $\beta$  is a real number satisfying  $\beta < \frac{1}{2} + \alpha$ , then the *controllability map*  $\mathcal{B} : L_2([0, \infty), U) \rightarrow Z_\beta$  given by

$$\mathcal{B}u = \int_0^\infty T(t)Bu(t)dt \tag{2.15}$$

is a well-defined bounded linear map, that is,  $B$  is an admissible input operator (see Weiss [86]) for the semigroup  $\{T(t)\}_{t \geq 0}$  on the Hilbert space  $Z_\beta$ , where  $\beta < \frac{1}{2} + \alpha$ . Moreover, in part 2. we show that if  $\beta$  and  $\gamma$  are real numbers satisfying  $\beta - \gamma < 1 + 2\alpha$ ,  $\beta < 1 + \alpha$ ,  $\gamma > -(1 + \alpha)$ , then the *controllability Gramian*  $L_B : Z'_\gamma \rightarrow Z_\beta$  given by

$$L_B z = \int_0^\infty T(t)BB'T(t)'z dt \tag{2.16}$$

	Curtain and Ichikawa [14]	Present approach
1.	They considered $A$ to be a uniformly strongly elliptic operator of order two with smooth coefficients defined on a bounded open domain $\Omega$ of $\mathbb{R}^d$ . Thus $-A$ is the infinitesimal generator of an exponentially stable, analytic, <i>compact</i> semigroup on $Z$ .	Here we simply assume that $-A$ is the infinitesimal generator of an exponentially stable, analytic semigroup on $Z$ .
2.	$B \in \mathcal{L}(U, Z_\alpha)$ , where $-1 < \alpha < -\frac{3}{4}$ .	$B \in \mathcal{L}(U, Z_\alpha)$ , where $-1 < \alpha \leq 0$ .
3.	$U = L_2(\partial\Omega)$ . Thus $U$ is <i>infinite</i> -dimensional, unless $\Omega$ is, for example, $(0, 1)$ , in which case $U$ can be identified with $\mathbb{C}^2$ .	$U = \mathbb{C}^m$

Table 2.1: Differences in assumptions.

is a well-defined bounded linear map from  $Z'_\gamma$  to  $Z_\beta$ .

**Lemma 2.4.1**  $\mathcal{B}$  and  $L_B$  given by (2.15) and (2.16) are well-defined and have the following properties:

1.  $\mathcal{B} \in \mathcal{L}(L_2([0, \infty), U), Z_\beta)$ , and  $\mathcal{B}' \in \mathcal{L}(Z'_{-\beta}, L_2([0, \infty), U))$ , for all  $\beta$  satisfying  $\beta < \frac{1}{2} + \alpha$ .
2.  $L_B \in \mathcal{L}(Z'_\gamma, Z_\beta)$  for all  $\gamma$  and  $\beta$  satisfying  $\beta - \gamma < 1 + 2\alpha$ ,  $\beta < 1 + \alpha$ ,  $\gamma > -(1 + \alpha)$ .
3.  $L_B = \mathcal{B}\mathcal{B}' \in \mathcal{L}(Z'_\gamma, Z_\beta)$  for  $\beta < \frac{1}{2} + \alpha$  and  $\gamma > -\frac{1}{2} - \alpha$  and it is self-dual in  $\mathcal{L}(Z'_{-\beta}, Z_\beta)$  for  $\beta < \frac{1}{2} + \alpha$ .
4.  $L_B \in \mathcal{L}(Z'_{-\beta}, Z_{\beta+1}) \cap \mathcal{L}(Z'_{-\beta-1}, Z_\beta)$  is a solution of the Lyapunov equation

$$AL_B z + L_B A' z = \mathcal{B}\mathcal{B}' z \quad \text{in } Z_\beta \quad (2.17)$$

for any  $z \in Z'_{-\beta}$ , where  $-1 < \beta < \alpha$ .

**Proof**

1. Let  $u \in L_2([0, \infty), U)$ . Since for  $t > 0$ ,  $T(t)Z_\alpha \subset \cap_{\beta \in \mathbb{R}} Z_\beta$ , it follows that  $T(t)Bu(t) \in Z_\beta$  for all  $\beta \in \mathbb{R}$ . Next we show that  $A^\beta T(\cdot)Bu(\cdot) \in L_1([0, \infty), Z)$  if  $\beta < \frac{1}{2} + \alpha$ :

$$\begin{aligned} & \int_0^\tau \|A^\beta T(t)Bu(t)\|_Z dt \\ &= \int_0^\tau \|A^{\beta-\alpha} T(t)\|_{\mathcal{L}(Z)} \|A^\alpha B\|_{\mathcal{L}(U,Z)} \|u(t)\|_U dt \\ &\leq \left[ \int_0^\tau \|A^{\beta-\alpha} T(t)\|_{\mathcal{L}(Z)}^2 dt \right]^{\frac{1}{2}} \|u\|_{L_2([0,\infty),U)} \|A^\alpha B\|_{\mathcal{L}(U,Z)}. \end{aligned}$$

**Case 1:**  $\beta - \alpha \geq 0$ . In this case,

$$\begin{aligned} & \int_0^\tau \|A^\beta T(t)Bu(t)\|_Z dt \\ &\leq \left[ \int_0^\tau \|A^{\beta-\alpha} T(t)\|_{\mathcal{L}(Z)}^2 dt \right]^{\frac{1}{2}} \|u\|_{L_2([0,\infty),U)} \|A^\alpha B\|_{\mathcal{L}(U,Z)} \\ &\leq \left[ \int_0^\tau \frac{K_1^2 e^{-2\epsilon t}}{t^{2(\beta-\alpha)}} dt \right]^{\frac{1}{2}} \|u\|_{L_2([0,\infty),U)} \|A^\alpha B\|_{\mathcal{L}(U,Z)}. \end{aligned}$$

For  $\beta$  satisfying  $2(\beta - \alpha) < 1$ , we have  $\int_0^\infty \frac{e^{-2\epsilon t}}{t^{2(\beta-\alpha)}} dt < \infty$ , and so

$$\int_0^\tau \|A^\beta T(t)Bu(t)\|_Z dt \leq \left[ \int_0^\infty \frac{K_1^2 e^{-2\epsilon t}}{t^{2(\beta-\alpha)}} dt \right]^{\frac{1}{2}} \|u\|_{L_2([0,\infty),U)} \|A^\alpha B\|_{\mathcal{L}(U,Z)}. \quad (2.18)$$

**Case 2:**  $\beta - \alpha < 0$ . In this case,

$$\begin{aligned} & \int_0^\tau \|A^\beta T(t)Bu(t)\|_Z dt \\ &\leq \left[ \int_0^\tau \|A^{\beta-\alpha} T(t)\|_{\mathcal{L}(Z)}^2 dt \right]^{\frac{1}{2}} \|u\|_{L_2([0,\infty),U)} \|A^\alpha B\|_{\mathcal{L}(U,Z)} \\ &\leq \left[ \int_0^\tau K_2^2 e^{-2\epsilon t} dt \right]^{\frac{1}{2}} \|u\|_{L_2([0,\infty),U)} \|A^\alpha B\|_{\mathcal{L}(U,Z)}. \end{aligned}$$

But  $\int_0^\infty e^{-2\epsilon t} dt < \infty$ , and so

$$\int_0^\tau \|A^\beta T(t)Bu(t)\|_Z dt \leq \left[ \int_0^\infty K_2^2 e^{-2\epsilon t} dt \right]^{\frac{1}{2}} \|u\|_{L_2([0,\infty),U)} \|A^\alpha B\|_{\mathcal{L}(U,Z)}. \quad (2.19)$$

From the above two cases, it follows that  $A^\beta T(\cdot)Bu(\cdot) \in L_1([0, \infty), Z)$ . Consequently, (see for example Theorem A.5.23, page 628, Curtain and Zwart [27]: with  $f, A, D(A), Z_1, Z_2$  and  $Af$  in the theorem replaced by  $T(\cdot)Bu(\cdot), A^\beta, Z_\beta, Z_\beta, Z$  and  $A^\beta T(\cdot)Bu(\cdot)$ , respectively)

$$A^\beta \int_0^\infty T(t)Bu(t)dt = \int_0^\infty A^\beta T(t)Bu(t)dt,$$

for every  $\beta < \frac{1}{2} + \alpha$ . Moreover, from (2.18) and (2.19) above, there exists a  $M > 0$  such that

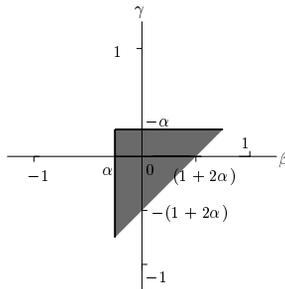
$$\left\| \int_0^\infty T(t)Bu(t)dt \right\|_{Z_\beta} = \left\| A^\beta \int_0^\infty T(t)Bu(t)dt \right\|_Z \leq M \|u\|_{L_2([0, \infty), U)}$$

and so  $B \in \mathcal{L}(L_2([0, \infty), U), Z_\beta)$ . Consequently, we also obtain that  $B' \in \mathcal{L}(Z'_{-\beta}, L_2([0, \infty), U))$  for  $\beta < \frac{1}{2} + \alpha$ .

2. From property (4), for every  $t > 0$  we have  $T(t)BB'T(t)'Z'_\gamma \subset Z_\beta$ , for any real  $\beta$  and  $\gamma$ . We first investigate the values of  $\gamma$  and  $\beta$  for which  $A^\beta T(t)BB'T(t)'(A')^{-\gamma}z \in L_1([0, \infty), Z)$  for  $z \in Z' = Z$ . For  $z \in Z$ , we have

$$\begin{aligned} & \int_0^\tau \|A^\beta T(t)BB'T(t)'(A')^{-\gamma}z\|_Z dt \\ &= \int_0^\tau \|A^{\beta-\alpha}T(t)A^\alpha BB'(A')^\alpha T(t)'(A')^{-\alpha-\gamma}z\|_Z dt \\ &= \int_0^\tau \|A^{\beta-\alpha}T(t)\|_{\mathcal{L}(Z)} \|A^\alpha B\|_{\mathcal{L}(U,Z)} \|B'(A')^\alpha\|_{\mathcal{L}(Z,U)} \\ & \quad \|T(t)'(A')^{-\alpha-\gamma}\|_{\mathcal{L}(Z)} \|z\|_Z dt. \end{aligned}$$

**Case 1:** Suppose that  $\beta - \alpha \geq 0, -\alpha - \gamma \geq 0$ .



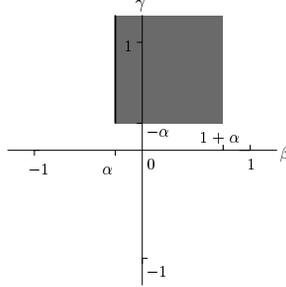
Then we have for  $z \in Z$ ,

$$\int_0^\tau \|A^\beta T(t)BB'T(t)'(A')^{-\gamma}z\|_Z dt$$

$$\begin{aligned} &\leq \left[ \int_0^\tau \frac{K_1^2 e^{-2\epsilon t}}{t^{\beta-2\alpha-\gamma}} dt \right] \|A^\alpha B\|_{\mathcal{L}(U,Z)} \|B'(A')^\alpha\|_{\mathcal{L}(Z,U)} \|z\|_Z \\ &\leq M_1 \|z\|_Z, \end{aligned}$$

where  $M_1$  is a constant independent of  $\tau$ . Thus  $A^\beta T(t) B B' T(t)' (A')^{-\gamma} z \in L_1([0, \infty), Z)$  for  $z \in Z$  and  $\beta - 2\alpha - \gamma < 1$ .

**Case 2:** Suppose that  $\beta - \alpha \geq 0$ ,  $-\alpha - \gamma < 0$ .

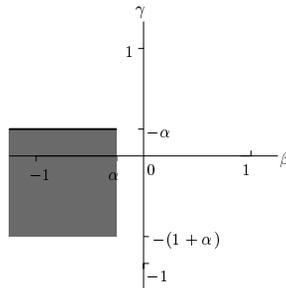


Then we have for  $z \in Z$ ,

$$\begin{aligned} &\int_0^\tau \|A^\beta T(t) B B' T(t)' (A')^{-\gamma} z\|_Z dt \\ &\leq \left[ \int_0^\tau \frac{K_1 K_2 e^{-2\epsilon t}}{t^{\beta-\alpha}} dt \right] \|A^\alpha B\|_{\mathcal{L}(U,Z)} \|B'(A')^\alpha\|_{\mathcal{L}(Z,U)} \|z\|_Z \\ &\leq M_2 \|z\|_Z, \end{aligned}$$

where  $M_2$  is a constant independent of  $\tau$ . Thus  $A^\beta T(t) B B' T(t)' (A')^{-\gamma} z \in L_1([0, \infty), Z)$  for  $z \in Z$  and  $\beta - \alpha < 1$ .

**Case 3:** Suppose that  $\beta - \alpha < 0$ ,  $-\alpha - \gamma \geq 0$ .



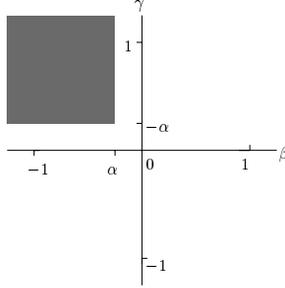
Then we have for  $z \in Z$ ,

$$\int_0^\tau \|A^\beta T(t) B B' T(t)' (A')^{-\gamma} z\|_Z dt$$

$$\begin{aligned}
&\leq \left[ \int_0^\tau \frac{K_1 K_2 e^{-2\epsilon t}}{t^{-\alpha-\gamma}} dt \right] \|A^\alpha B\|_{\mathcal{L}(U,Z)} \|B'(A')^\alpha\|_{\mathcal{L}(Z,U)} \|z\|_Z \\
&\leq M_3 \|z\|_Z,
\end{aligned}$$

where  $M_3$  is a constant independent of  $\tau$ . Thus  $A^\beta T(t) B B' T(t)' (A')^{-\gamma} z \in L_1([0, \infty), Z)$  for  $z \in Z$  and  $-\alpha - \gamma < 1$ .

**Case 4:** Suppose that  $\beta - \alpha < 0$ ,  $-\alpha - \gamma \geq 0$ .



Then we have for  $z \in Z$ ,

$$\begin{aligned}
&\int_0^\tau \|A^\beta T(t) B B' T(t)' (A')^{-\gamma} z\|_Z dt \\
&\leq \left[ \int_0^\tau K_2^2 e^{-2\epsilon t} dt \right] \|A^\alpha B\|_{\mathcal{L}(U,Z)} \|B'(A')^\alpha\|_{\mathcal{L}(Z,U)} \|z\|_Z \\
&\leq M_4 \|z\|_Z,
\end{aligned}$$

where  $M_4$  is a constant independent of  $\tau$ . Thus  $A^\beta T(t) B B' T(t)' (A')^{-\gamma} z \in L_1([0, \infty), Z)$  for  $z \in Z$ .

If  $z \in Z'_\gamma$ , then  $(A')^\gamma z \in Z$  and applying the above results,

$$\begin{aligned}
\int_0^\infty \|A^\beta T(t) B B' T(t)' z\|_Z dt &\leq \int_0^\infty \|A^\beta T(t) B B' T(t)' (A')^{-\gamma} (A')^\gamma z\|_Z dt \\
&\leq M \|(A')^\gamma z\|_Z = M \|z\|_{Z'_\gamma},
\end{aligned}$$

for some constant  $M > 0$ . So if  $z \in Z'_\gamma$ , then

$$\begin{aligned}
\|L_B z\|_{Z_\beta} &= \left\| A^\beta \int_0^\infty T(t) B B' T(t)' z dt \right\|_Z \\
&= \left\| \int_0^\infty A^\beta T(t) B B' T(t)' z dt \right\|_Z \\
&\quad \text{(using Theorem A.5.23, page 628, [27])} \\
&\leq \int_0^\infty \|A^\beta T(t) B B' T(t)' z\|_Z dt \\
&\leq M \|z\|_{Z'_\gamma},
\end{aligned}$$

for  $\beta - 2\alpha - \gamma < 1$ ,  $\beta < 1 + \alpha$ ,  $\gamma > -(1 + \alpha)$ . We depict this region (shaded) in Figure 2.4.

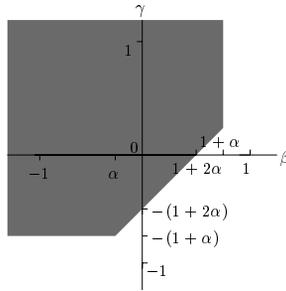


Figure 2.4: The feasible region of parameter values for  $L_B$ .

We note that the point  $(1 + 2\alpha, 0)$  lies to the right of the point  $(\alpha, 0)$  since  $1 + \alpha > 0$ . However, whether the point  $(1 + 2\alpha, 0)$  lies to the right or to the left or coincides with the origin depends on the value of  $\alpha$ :

$$\begin{aligned} -1 < \alpha < -\frac{1}{2} & : \text{ left,} \\ \alpha = -\frac{1}{2} & : \text{ coincides,} \\ -\frac{1}{2} < \alpha \leq 0 & : \text{ right.} \end{aligned}$$

(In Figure 2.4, we have shown the case when  $\alpha$  satisfies  $-\frac{1}{2} < \alpha \leq 0$ .)

3. That  $\mathcal{B}\mathcal{B}' \in \mathcal{L}(Z'_\gamma, Z_\beta)$  for  $\beta < \frac{1}{2} + \alpha$  and  $\gamma > -\frac{1}{2} - \alpha$  and is self-dual in  $\mathcal{L}(Z'_{-\beta}, Z_\beta)$  for  $\beta < \frac{1}{2} + \alpha$  is a consequence of part 1 above.

So it remains to prove the equality  $L_B = \mathcal{B}\mathcal{B}' \in \mathcal{L}(Z'_\gamma, Z_\beta)$  for  $\beta < \frac{1}{2} + \alpha$  and  $\gamma > -\frac{1}{2} - \alpha$ :

It can be easily shown that if  $z \in Z'_{-\beta}$  for  $\beta < \frac{1}{2} + \alpha$ , then

$$(B'z)(t) = B'T(t)'z, \quad t \geq 0.$$

Thus if  $z \in Z'_\gamma$  for  $\gamma > -\frac{1}{2} - \alpha$ , then  $B'z = B'T(\cdot)'z \in L_2([0, \infty), U)$ , and since  $B \in \mathcal{L}(L_2([0, \infty), U), Z_\beta)$  for  $\beta < \frac{1}{2} + \alpha$ ,

$$\mathcal{B}\mathcal{B}'z = \int_0^\infty T(t)B\mathcal{B}'z dt = \int_0^\infty T(t)B\mathcal{B}'T(t)'z dt \in Z_\beta.$$

But since  $\beta - \gamma < (\frac{1}{2} + \alpha) + (\frac{1}{2} + \alpha) = 1 + 2\alpha$ ,  $\beta < \frac{1}{2} + \alpha < 1 + \alpha$ ,  $\gamma > -\frac{1}{2} - \alpha >$

$-1 - \alpha$ , it follows from part 2 that

$$\int_0^\infty T(t)BB'T(t)'zdt = L_B z \in Z_\beta.$$

Thus  $L_B = BB' \in \mathcal{L}(Z'_\gamma, Z_\beta)$  for  $\beta < \frac{1}{2} + \alpha$  and  $\gamma > -\frac{1}{2} - \alpha$ .

4. From A2,  $B'(A')^\beta$  belongs to  $\mathcal{L}(Z, U)$  for  $\beta$  satisfying  $-1 < \beta \leq \alpha$ . Since  $-A$  is the infinitesimal generator of an exponentially stable semigroup on the Hilbert space  $Z$ , it follows from the standard theory for Lyapunov equations (see for example Theorem 4.1.23, page 140, Curtain and Zwart [27]), applied to  $\Sigma(-A, B'(A')^\beta, -)$  with state space  $Z$ , that there exists a unique self-adjoint solution  $P \in \mathcal{L}(Z)$  of the operator Lyapunov equation

$$\langle A'z_1, Pz_2 \rangle_Z + \langle Pz_1, A'z_2 \rangle_Z = \langle B'(A')^\beta z_1, B'(A')^\beta z_2 \rangle_U,$$

where  $z_1$  and  $z_2$  belong to  $D(A')$  and  $P$  is given by

$$Pz = \int_0^\infty T(t)A^\beta BB'(A')^\beta T(t)'zdt \text{ for } z \in Z.$$

If  $w_1$  and  $w_2$  belong to  $Z'_{1-\beta}$ , then  $(A')^{-\beta}w_1$  and  $(A')^{-\beta}w_2$  belong to  $D(A')$ . Consequently, we have

$$\langle A'(A')^{-\beta}w_1, P(A')^{-\beta}w_2 \rangle_Z + \langle P(A')^{-\beta}w_1, A'(A')^{-\beta}w_2 \rangle_Z = \langle B'w_1, B'w_2 \rangle_U. \quad (2.20)$$

We now proceed to relate  $P$  to  $L_B$  defined in part 2. If  $\beta < \alpha$ , we have, using part 2, that  $L_B \in \mathcal{L}(Z'_{-\beta}, Z_{\beta+1})$  and so  $AL_B \in \mathcal{L}(Z'_{-\beta}, Z_\beta)$ . If  $w \in Z'_{1-\beta}$ , then  $w \in Z'_{-\beta}$ , since  $Z'_{1-\beta} \subset Z'_{-\beta}$ . If  $\langle \cdot, \cdot \rangle_{\langle Z'_{-\beta}, Z_\beta \rangle}$  denotes the duality pairing between  $Z'_{-\beta}$  and  $Z_\beta$ , then for  $w_1$  and  $w_2$  in  $Z'_{1-\beta}$ , with  $-1 < \beta < \alpha$ , we have

$$\begin{aligned} \langle w_1, AL_B w_2 \rangle_{\langle Z'_{-\beta}, Z_\beta \rangle} &= \left\langle w_1, A \int_0^\infty T(t)BB'T(t)'w_2 dt \right\rangle_{\langle Z'_{-\beta}, Z_\beta \rangle} \\ &= \left\langle A'w_1, \int_0^\infty T(t)BB'T(t)'w_2 dt \right\rangle_{\langle Z'_{-\beta-1}, Z_{\beta+1} \rangle}, \end{aligned}$$

where we have used  $A \in \mathcal{L}(Z_{\beta+1}, Z_\beta)$  and  $A' \in \mathcal{L}(Z'_{-\beta}, Z'_{-\beta-1})$ . Since  $w_1 \in Z'_{1-\beta} = D((A')^{1-\beta})$ ,  $A'w_1$  belongs to  $D((A')^{-\beta}) = Z'_{-\beta}$ . From part 2,  $L_B \in \mathcal{L}(Z'_{-\beta}, Z_{\beta+1})$  and since  $w_2$  belongs to  $Z'_{1-\beta} \subset Z'_{-\beta}$ , it follows that  $L_B w_2 \in Z_{\beta+1} \subset Z_\beta$  and so

$$\begin{aligned} \langle w_1, AL_B w_2 \rangle_{\langle Z'_{-\beta}, Z_\beta \rangle} &= \left\langle A'w_1, \int_0^\infty T(t)BB'T(t)'w_2 dt \right\rangle_{\langle Z'_{-\beta-1}, Z_{\beta+1} \rangle} \\ &= \left\langle A'w_1, \int_0^\infty T(t)BB'T(t)'w_2 dt \right\rangle_{\langle Z'_{-\beta}, Z_\beta \rangle}. \end{aligned}$$

Hence

$$\langle w_1, AL_B w_2 \rangle_{\langle Z'_{-\beta}, Z_\beta \rangle} = \int_0^\infty \langle A' w_1, T(t) B B' T(t)' w_2 \rangle_{\langle Z'_{-\beta}, Z_\beta \rangle} dt,$$

since  $A' w_1 \in D((A')^{-\beta})$  and  $T(t) B B' T(t)' w_2 \in Z_\beta$ . Consequently,

$$\begin{aligned} & \langle w_1, AL_B w_2 \rangle_{\langle Z'_{-\beta}, Z_\beta \rangle} \\ &= \int_0^\infty \langle A' w_1, A^{-\beta} T(t) A^\beta B B' (A')^\beta T(t)' (A')^{-\beta} w_2 \rangle_{\langle Z'_{-\beta}, Z_\beta \rangle} dt \\ &= \int_0^\infty \langle A' (A')^{-\beta} w_1, T(t) A^\beta B B' (A')^\beta T(t)' (A')^{-\beta} w_2 \rangle_Z dt, \end{aligned}$$

since  $A^{-\beta} \in \mathcal{L}(Z, Z_\beta)$  and  $A' (A')^{-\beta} w_1 \in Z$ . As a result we obtain that

$$\begin{aligned} & \langle w_1, AL_B w_2 \rangle_{\langle Z'_{-\beta}, Z_\beta \rangle} \\ &= \left\langle A' (A')^{-\beta} w_1, \int_0^\infty T(t) A^\beta B B' (A')^\beta T(t)' (A')^{-\beta} w_2 dt \right\rangle_Z \\ &= \langle A' (A')^{-\beta} w_1, P(A')^{-\beta} w_2 \rangle_Z. \end{aligned} \quad (2.21)$$

Similarly, it can be shown that for  $w_1$  and  $w_2$  in  $Z'_{1-\beta}$ , we have

$$\langle w_1, L_B A' w_2 \rangle_{\langle Z'_{-\beta}, Z_\beta \rangle} = \langle P(A')^{-\beta} w_1, A' (A')^{-\beta} w_2 \rangle_Z. \quad (2.22)$$

(Here, using part 2, we consider  $L_B \in \mathcal{L}(Z'_{-\beta-1}, Z_\beta)$  for  $\beta < \alpha$ ; we have  $L_B A' \in \mathcal{L}(Z'_{-\beta}, Z_\beta)$ .)

So by substituting (2.21) and (2.22) in (2.20), we see that  $L_B$  satisfies

$$\langle z_1, AL_B z_2 \rangle_{\langle Z'_{-\beta}, Z_\beta \rangle} + \langle z_1, L_B A' z_2 \rangle_{\langle Z'_{-\beta}, Z_\beta \rangle} = \langle z_1, BB' z_2 \rangle_{\langle Z'_{-\beta}, Z_\beta \rangle}$$

for  $z_1$  and  $z_2$  in  $Z'_{-\beta+1} (\subset Z'_{-\beta})$ . But since  $Z'_{-\beta+1}$  is dense in  $Z'_{-\beta}$ , and  $AL_B$ ,  $L_B A'$  and  $BB' \in \mathcal{L}(Z'_{-\beta}, Z_\beta)$ , it follows that

$$\langle z_1, AL_B z_2 \rangle_{\langle Z'_{-\beta}, Z_\beta \rangle} + \langle z_1, L_B A' z_2 \rangle_{\langle Z'_{-\beta}, Z_\beta \rangle} = \langle z_1, BB' z_2 \rangle_{\langle Z'_{-\beta}, Z_\beta \rangle}$$

holds for  $z_1$  and  $z_2$  in  $Z'_{-\beta}$ . ■

Next we derive some properties of the *observability map*  $\mathcal{C} : Z \rightarrow L_2([0, \infty), Y)$  given by

$$\mathcal{C}z = CT(\cdot)z, \quad (2.23)$$

and the *observability Gramian*  $L_C : Z \rightarrow Z$  given by

$$L_C z = \int_0^\infty T(t)' C' C T(t) z dt. \quad (2.24)$$

Owing to the smoothing property of  $\{T(t)\}_{t \geq 0}$ , both  $\mathcal{C}$  and  $L_C$  inherit some smoothness too.

**Lemma 2.4.2**  $\mathcal{C}$  and  $L_C$  defined by (2.23) and (2.24) have the following properties:

1.  $\mathcal{C} \in \mathcal{L}(Z_\beta, L_2([0, \infty), Y))$  and  $\mathcal{C}' \in \mathcal{L}(L_2([0, \infty), Y), Z'_{-\beta})$ , for all  $\beta$  satisfying  $\beta > -\frac{1}{2}$ .
2.  $L_C \in \mathcal{L}(Z_\beta, Z'_\gamma)$  for all  $\gamma$  and  $\beta$  satisfying  $\gamma - \beta < 1$ ,  $\beta > -1$ ,  $\gamma < 1$ .
3.  $L_C = \mathcal{C}'\mathcal{C} \in \mathcal{L}(Z_\beta, Z'_\gamma)$  for  $\beta > -\frac{1}{2}$  and  $\gamma < \frac{1}{2}$  and it is self-dual in  $\mathcal{L}(Z_\beta, Z'_{-\beta})$  for  $\beta > -\frac{1}{2}$ .
4. If  $\epsilon > 0$ , then  $L_C \in \mathcal{L}(Z_\epsilon, Z'_{-\epsilon+1}) \cap \mathcal{L}(Z_{\epsilon-1}, Z'_{-\epsilon})$  is a solution of the Lyapunov equation

$$A' L_C z + L_C A z = \mathcal{C}' \mathcal{C} z \quad \text{in } Z'_{-\epsilon} \quad (2.25)$$

for any  $z \in Z_\epsilon$ .

### Proof

1. If  $z \in Z_\beta$ , then

$$\begin{aligned} \int_0^\tau \|CT(t)z\|_Y^2 dt &\leq \int_0^\tau \|C\|_{\mathcal{L}(Z, Y)}^2 \|T(t)z\|_Z^2 dt \\ &= \|C\|_{\mathcal{L}(Z, Y)}^2 \int_0^\tau \|A^{-\beta} T(t) A^\beta z\|_Z^2 dt \\ &\leq \|C\|_{\mathcal{L}(Z, Y)}^2 \int_0^\tau \|A^{-\beta} T(t)\|_{\mathcal{L}(Z)}^2 \|A^\beta z\|_Z^2 dt \\ &= \|C\|_{\mathcal{L}(Z, Y)}^2 \|z\|_{Z_\beta}^2 \int_0^\tau \|A^{-\beta} T(t)\|_{\mathcal{L}(Z)}^2 dt. \end{aligned}$$

Case 1:  $-\frac{1}{2} < \beta \leq 0$ . If  $z \in Z_\beta$ , then

$$\begin{aligned} \int_0^\tau \|CT(t)z\|_Y^2 dt &\leq \|C\|_{\mathcal{L}(Z, Y)}^2 \|z\|_{Z_\beta}^2 \int_0^\tau \frac{K_1^2 e^{-2\epsilon t}}{t^{-2\beta}} dt \\ &\leq \|C\|_{\mathcal{L}(Z, Y)}^2 \|z\|_{Z_\beta}^2 \int_0^\infty \frac{K_1^2 e^{-2\epsilon t}}{t^{-2\beta}} dt \\ &\leq M_1^2 \|z\|_{Z_\beta}^2, \end{aligned}$$

for some constant  $M_1 > 0$  (independent of  $\tau$ ), since  $-2\beta < 1$ .

**Case 2:**  $0 < \beta$ . If  $z \in Z_\beta$ , then

$$\begin{aligned} \int_0^\tau \|CT(t)z\|_Y^2 dt &\leq \|C\|_{\mathcal{L}(Z,Y)}^2 \|z\|_{Z_\beta}^2 \int_0^\tau K_2^2 e^{-2\epsilon t} dt \\ &\leq \|C\|_{\mathcal{L}(Z,Y)}^2 \|z\|_{Z_\beta}^2 \int_0^\infty K_2^2 e^{-2\epsilon t} dt \\ &\leq M_2^2 \|z\|_{Z_\beta}^2, \end{aligned}$$

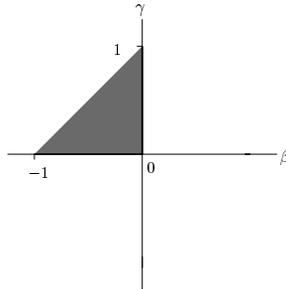
for some constant  $M_2 > 0$  (independent of  $\tau$ ).

Thus we have  $CT(\cdot)z \in L_2([0, \infty), Y)$  and moreover from the above we obtain  $\|Cz\|_{L_2([0, \infty), Y)} \leq M \|z\|_{Z_\beta}$ , for some  $M > 0$ . So it follows that  $C \in \mathcal{L}(Z_\beta, L_2([0, \infty), Y))$  for  $\beta > -\frac{1}{2}$  and  $C' \in \mathcal{L}(L_2([0, \infty), Y), Z'_{-\beta})$  for  $\beta > -\frac{1}{2}$ .

2. From property (4), for every  $t > 0$  we have  $T(t)'C'CT(t)Z_\beta \subset Z'_{-\gamma}$ , for any real  $\beta$  and  $\gamma$ . We now investigate the values of  $\gamma$  and  $\beta$  for which  $(A')^\gamma T(t)'C'CT(t)A^{-\beta}z \in L_1([0, \infty), Z')$  for  $z \in Z$ . If  $z \in Z$ ,

$$\begin{aligned} &\int_0^\tau \|(A')^\gamma T(t)'C'CT(t)A^{-\beta}z\|_{Z'} dt \\ &= \int_0^\tau \|(A')^\gamma T(t)'\|_{\mathcal{L}(Z')} \|C'\|_{\mathcal{L}(Y,Z')} \|C\|_{\mathcal{L}(Z,Y)} \|T(t)A^{-\beta}\|_{\mathcal{L}(Z)} \|z\|_Z dt \\ &\leq \|C\|_{\mathcal{L}(Z,Y)}^2 \|z\|_Z \int_0^\tau \|(A')^\gamma T(t)'\|_{\mathcal{L}(Z')} \|T(t)A^{-\beta}\|_{\mathcal{L}(Z)} dt. \end{aligned}$$

**Case 1:**  $\gamma \geq 0, \beta \leq 0$ .

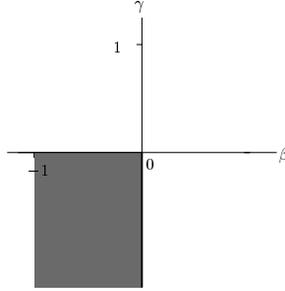


If  $z \in Z$ ,

$$\begin{aligned} \int_0^\tau \|(A')^\gamma T(t)'C'CT(t)A^{-\beta}z\|_{Z'} dt &\leq \|C\|_{\mathcal{L}(Z,Y)}^2 \|z\|_Z \int_0^\tau \frac{K_1^2 e^{-2\epsilon t}}{t^{\gamma-\beta}} dt \\ &\leq \|C\|_{\mathcal{L}(Z,Y)}^2 \|z\|_Z \int_0^\infty \frac{K_1^2 e^{-2\epsilon t}}{t^{\gamma-\beta}} dt \\ &\leq M_1 \|z\|, \end{aligned}$$

for some constant  $M_1 > 0$  (independent of  $\tau$ ) if  $\gamma - \beta < 1$ .

**Case 2:**  $\gamma < 0, \beta \leq 0$ .

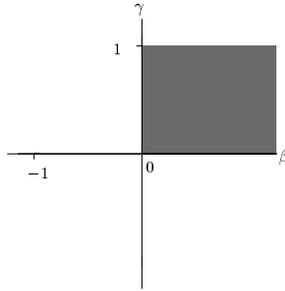


If  $z \in Z$ ,

$$\begin{aligned} \int_0^\tau \|(A')^\gamma T(t)' C' C T(t) A^{-\beta} z\|_Z dt &\leq \|C\|_{\mathcal{L}(Z, Y)}^2 \|z\|_Z \int_0^\tau \frac{K_1 K_2 e^{-2\epsilon t}}{t^{-\beta}} dt \\ &\leq \|C\|_{\mathcal{L}(Z, Y)}^2 \|z\|_Z \int_0^\infty \frac{K_1 K_2 e^{-2\epsilon t}}{t^{-\beta}} dt \\ &\leq M_2 \|z\|, \end{aligned}$$

for some constant  $M_2 > 0$  (independent of  $\tau$ ) if  $-\beta < 1$ , that is, if  $\beta > -1$ .

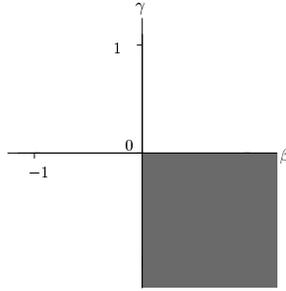
**Case 3:**  $\gamma \geq 0, \beta > 0$ .



If  $z \in Z$ ,

$$\begin{aligned} \int_0^\tau \|(A')^\gamma T(t)' C' C T(t) A^{-\beta} z\|_Z dt &\leq \|C\|_{\mathcal{L}(Z, Y)}^2 \|z\|_Z \int_0^\tau \frac{K_1 K_2 e^{-2\epsilon t}}{t^\gamma} dt \\ &\leq \|C\|_{\mathcal{L}(Z, Y)}^2 \|z\|_Z \int_0^\infty \frac{K_1 K_2 e^{-2\epsilon t}}{t^\gamma} dt \\ &\leq M_3 \|z\|, \end{aligned}$$

for some constant  $M_3 > 0$  (independent of  $\tau$ ) if  $\gamma < 1$ .



**Case 4:**  $\gamma < 0, \beta > 0$ .

If  $z \in Z$ ,

$$\begin{aligned} \int_0^\tau \|(A')^\gamma T(t)'C'CT(t)A^{-\beta}z\|_Z dt &\leq \|C\|_{\mathcal{L}(Z,Y)}^2 \|z\|_Z \int_0^\tau K_2^2 e^{-2\epsilon t} dt \\ &\leq \|C\|_{\mathcal{L}(Z,Y)}^2 \|z\|_Z \int_0^\infty K_2^2 e^{-2\epsilon t} dt \\ &\leq M_4 \|z\|, \end{aligned}$$

for some constant  $M_4 > 0$  (independent of  $\tau$ ).

So  $(A')^\gamma T(t)'C'CT(t)A^{-\beta}z \in L_1([0, \infty), Z)$  for  $z \in Z$ . If  $z \in Z_\beta$ , then  $A^\beta z \in Z_\beta$  and using the above, we obtain for  $\gamma - \beta < 1, \gamma < 1, \beta > -1$ ,

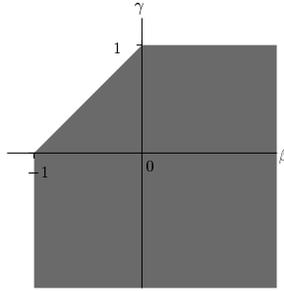
$$\begin{aligned} \|L_C z\|_{Z'_\gamma} &= \left\| (A')^\gamma \int_0^\infty T(t)'C'CT(t)z dt \right\|_Z \\ &= \left\| \int_0^\infty (A')^\gamma T(t)'C'CT(t)z dt \right\|_Z \\ &= \left\| \int_0^\infty (A')^\gamma T(t)'C'CT(t)A^{-\beta}A^\beta z dt \right\|_Z \\ &\leq \int_0^\infty \|(A')^\gamma T(t)'C'CT(t)A^{-\beta}A^\beta z\| dt \\ &\leq M \|A^\beta z\|_Z = M \|z\|_{Z_\beta}, \end{aligned}$$

for some constant  $M > 0$ . We depict this region (shaded) in Figure 2.5.

3. That  $C'C \in \mathcal{L}(Z_\beta, Z'_\gamma)$  for  $\beta > -\frac{1}{2}$  and  $\gamma < \frac{1}{2}$  and is self-dual in  $\mathcal{L}(Z_\beta, Z'_{-\beta})$  for  $\beta > -\frac{1}{2}$  is a consequence of part 1 above.

We only need to prove that the equality  $L_C = C'C \in \mathcal{L}(Z_\beta, Z'_\gamma)$  holds for  $\beta > -\frac{1}{2}$  and  $\gamma < \frac{1}{2}$ :

From 1,  $C' \in \mathcal{L}(L_2([0, \infty), Y), Z'_\gamma)$  for  $\gamma < \frac{1}{2}$  and it can be easily shown that

Figure 2.5: The feasible region of parameter values for  $L_C$ .

for any  $y \in L_2([0, \infty), Y)$

$$C'y = \int_0^\infty T(t)'C'y(t)dt \in Z'_\gamma.$$

If  $z$  with  $Z_\beta$  for  $\beta > -\frac{1}{2}$ , then since  $C \in \mathcal{L}(Z_\beta, L_2([0, \infty), Y))$ ,

$$(Cz)(\cdot) = CT(\cdot)z \in L_2([0, \infty), Y),$$

and so

$$C'Cz = C'(CT(\cdot)z) = \int_0^\infty T(t)'C'CT(t)zdt \in Z'_\gamma.$$

Now since  $\gamma - \beta < (\frac{1}{2}) - (-\frac{1}{2}) = 1$ ,  $\gamma < \frac{1}{2} < 1$ ,  $\beta > -\frac{1}{2} > -1$ , it follows from part 2 that

$$\int_0^\infty T(t)'C'CT(t)zdt = L_Cz \in Z'_\gamma.$$

So  $L_C = C'C \in \mathcal{L}(Z_\beta, Z'_\gamma)$  for  $\beta > -\frac{1}{2}$  and  $\gamma < \frac{1}{2}$ .

4.  $C \in \mathcal{L}(Z, Y)$  and so  $CA^{-\epsilon} \in \mathcal{L}(Z, Y)$  for  $\epsilon > 0$  (using Lemma 6.3, page 71, Pazy [63]). Since  $-A$  is the infinitesimal generator of an exponentially stable strongly continuous semigroup on the Hilbert space  $Z$ , it follows from the standard theory for Lyapunov equations (see for example Theorem 4.1.23, page 140, Curtain and Zwart [27]) applied to  $\Sigma(-A, -, CA^{-\epsilon})$  with state space  $Z$ , that there exists a unique self-adjoint solution  $Q$  of the operator Lyapunov equation

$$\langle Az_1, Qz_2 \rangle_Z + \langle Qz_1, Az_2 \rangle_Z = \langle CA^{-\epsilon}z_1, CA^{-\epsilon}z_2 \rangle_Y,$$

for all  $z_1$  and  $z_2$  in  $D(A)$  and  $Q$  is given by

$$Qz = \int_0^\infty T(t)'(A')^{-\epsilon}C'CA^{-\epsilon}T(t)zdt,$$

for all  $z \in Z$ . If  $w_1$  and  $w_2$  belong to  $Z_{1+\epsilon}$ , then  $A^\epsilon w_1$  and  $A^\epsilon w_2$  belong to  $D(A)$ . Consequently, we have

$$\langle AA^\epsilon w_1, QA^\epsilon w_2 \rangle_Z + \langle QA^\epsilon w_1, AA^\epsilon w_2 \rangle_Z = \langle Cw_1, Cw_2 \rangle_Y. \quad (2.26)$$

If  $\epsilon > 0$ , we have, using part 2, that  $L_C \in \mathcal{L}(Z_\epsilon, Z'_{-\epsilon+1})$  and so  $A'L_C \in \mathcal{L}(Z_\epsilon, Z'_{-\epsilon})$ . If  $w \in Z_{1+\epsilon}$ , then  $w \in Z_\epsilon$ , since  $Z_{1+\epsilon} \subset Z_\epsilon$ . If  $\langle \cdot, \cdot \rangle_{\langle Z_\epsilon, Z'_{-\epsilon} \rangle}$  denotes the duality pairing between  $Z_\epsilon$  and  $Z'_{-\epsilon}$ , with  $\epsilon > 0$ , then for  $w_1, w_2 \in Z_{1+\epsilon}$ , we have

$$\begin{aligned} \langle w_1, A'L_C w_2 \rangle_{\langle Z_\epsilon, Z'_{-\epsilon} \rangle} &= \left\langle w_1, A' \int_0^\infty T(t)' C' C T(t) w_2 dt \right\rangle_{\langle Z_\epsilon, Z'_{-\epsilon} \rangle} \\ &= \left\langle Aw_1, \int_0^\infty T(t)' C' C T(t) w_2 dt \right\rangle_{\langle Z_{\epsilon-1}, Z'_{1-\epsilon} \rangle}, \end{aligned}$$

since  $A' \in \mathcal{L}(Z'_{-\epsilon+1}, Z'_{-\epsilon})$  and  $A \in \mathcal{L}(Z_\epsilon, Z'_{\epsilon-1})$ . Since  $w_1 \in Z_{1+\epsilon} = D(A^{1+\epsilon})$ ,  $Aw_1$  belongs to  $D(A^\epsilon)$ . From part 2,  $L_C \in \mathcal{L}(Z_\epsilon, Z'_{-\epsilon+1})$  and since  $w_2 \in Z_{1+\epsilon} \subset Z_\epsilon$ ,  $L_C w_2 \in Z'_{-\epsilon+1} \subset Z'_{-\epsilon}$  and so

$$\begin{aligned} \langle w_1, A'L_C w_2 \rangle_{\langle Z_\epsilon, Z'_{-\epsilon} \rangle} &= \left\langle Aw_1, \int_0^\infty T(t)' C' C T(t) w_2 dt \right\rangle_{\langle Z_{\epsilon-1}, Z'_{1-\epsilon} \rangle} \\ &= \left\langle Aw_1, \int_0^\infty T(t)' C' C T(t) w_2 dt \right\rangle_{\langle Z_\epsilon, Z'_{-\epsilon} \rangle}. \end{aligned}$$

Since  $Aw_1 \in D(A^\epsilon)$  and  $T(t)' C' C T(t) w_2 \in Z'_{-\epsilon}$ ,

$$\langle w_1, A'L_C w_2 \rangle_{\langle Z_\epsilon, Z'_{-\epsilon} \rangle} = \int_0^\infty \langle Aw_1, T(t)' C' C T(t) w_2 \rangle_{\langle Z_\epsilon, Z'_{-\epsilon} \rangle} dt.$$

Consequently,

$$\begin{aligned} &\langle w_1, A'L_C w_2 \rangle_{\langle Z_\epsilon, Z'_{-\epsilon} \rangle} \\ &= \int_0^\infty \langle Aw_1, (A')^\epsilon T(t)' (A')^{-\epsilon} C' C A^{-\epsilon} T(t) A^\epsilon w_2 \rangle_{\langle Z_\epsilon, Z'_{-\epsilon} \rangle} dt \\ &= \int_0^\infty \langle AA^\epsilon w_1, T(t)' (A')^{-\epsilon} C' C A^{-\epsilon} T(t) A^\epsilon w_2 \rangle_Z dt \end{aligned}$$

since  $(A')^\epsilon \in \mathcal{L}(Z, Z'_{-\epsilon})$  and  $AA^\epsilon w_1 \in Z$ . As a result we obtain that

$$\begin{aligned} &\langle w_1, A'L_C w_2 \rangle_{\langle Z_\epsilon, Z'_{-\epsilon} \rangle} \\ &= \left\langle AA^\epsilon w_1, \int_0^\infty T(t)' (A')^{-\epsilon} C' C A^{-\epsilon} T(t) A^\epsilon w_2 dt \right\rangle_Z \\ &= \langle AA^\epsilon w_1, QA^\epsilon w_2 \rangle_Z. \end{aligned} \quad (2.27)$$

Similarly, it can be shown that for  $w_1$  and  $w_2$  in  $Z_{1+\epsilon}$ , we have

$$\langle w_1, L_C A w_2 \rangle_{\langle Z_\epsilon, Z'_{-\epsilon} \rangle} = \langle Q A^\epsilon w_1, A A^\epsilon w_2 \rangle_Z. \quad (2.28)$$

(Here, using part 2, we consider  $L_C \in \mathcal{L}(Z_{\epsilon-1}, Z'_{-\epsilon})$  for  $\epsilon > 0$ ; we have  $L_C A \in \mathcal{L}(Z_\epsilon, Z'_{-\epsilon})$ .)

So by substituting (2.27) and (2.28) in (2.26), we see that  $L_C$  satisfies

$$\langle z_1, A' L_C z_2 \rangle_{\langle Z_\epsilon, Z'_{-\epsilon} \rangle} + \langle z_1, L_C A z_2 \rangle_{\langle Z_\epsilon, Z'_{-\epsilon} \rangle} = \langle z_1, C' C z_2 \rangle_{\langle Z_\epsilon, Z'_{-\epsilon} \rangle}$$

for  $z_1$  and  $z_2$  in  $Z_{1+\epsilon} (\subset Z_\epsilon)$ . But since  $Z_{1+\epsilon}$  is dense in  $Z_\epsilon$ , and  $A' L_C$ ,  $L_C A$  and  $C' C \in \mathcal{L}(Z_\epsilon, Z'_{-\epsilon})$ , it follows that

$$\langle z_1, A' L_C z_2 \rangle_{\langle Z_\epsilon, Z'_{-\epsilon} \rangle} + \langle z_1, L_C A z_2 \rangle_{\langle Z_\epsilon, Z'_{-\epsilon} \rangle} = \langle z_1, C' C z_2 \rangle_{\langle Z_\epsilon, Z'_{-\epsilon} \rangle}$$

holds for  $z_1$  and  $z_2$  in  $Z_\epsilon$ . ■

**Remark:** We note that part 1 of Lemma 2.4.2 shows that  $C$  is an admissible observation operator for the semigroup  $\{T(t)\}_{t \geq 0}$  on the Hilbert space  $Z_\beta$ , for  $\beta > -\frac{1}{2}$ .

We now show that the system  $\Sigma$  given by the triple  $(-A, B, C)$  is a well-posed linear system on  $Z_\beta$  for any  $\beta \in \mathbb{R}$  satisfying  $-\frac{1}{2} < \beta < \frac{1}{2} + \alpha$ .

**Theorem 2.4.3** *Let  $\beta \in \mathbb{R}$  satisfy  $-\frac{1}{2} < \beta < \frac{1}{2} + \alpha$ . If  $A$ ,  $B$  and  $C$  satisfy the assumptions A1, A2 and A3 listed at the beginning of this section, then the system given by the triple  $(-A, B, C)$  is a well-posed linear system with input space  $U$ , state space  $Z_\beta$  and output space  $Y$ , and has a transfer function given by  $G(s) = C(sI + A)^{-1}B$ .*

**Proof** From part 1 of Theorem 2.4.1 and part 1 of Theorem 2.4.2, it follows that  $B$  and  $C$  are admissible input and output operators for the semigroup  $\{T(t)\}_{t \geq 0}$  with state space  $Z_\beta$ . We show that  $\Theta(s) = C(sI + A)^{-1}B$  is well-defined for  $s \in \rho(-A)$ .

1. Let  $s \in \rho(-A)$ . Firstly, if  $u \in U$ , then  $Bu \in Z_\alpha$  and since  $(sI + A)^{-1} \in \mathcal{L}(Z_\alpha, Z_{\alpha+1})$ , it follows that  $(sI + A)^{-1}Bu \in Z_{\alpha+1}$ . Furthermore, since  $\alpha > -1$ , it follows that  $(sI + A)^{-1}Bu \in Z$ . Hence  $C(sI + A)^{-1}Bu \in Y$ .

2. Furthermore, we have for  $u \in U$ ,

$$\|\Theta u\|_Y \leq \|C(sI + A)^{-1}Bu\|_Y$$

$$\begin{aligned}
&\leq \|C\|_{\mathcal{L}(Z_{\alpha+1}, Y)} \|(sI + A)^{-1}Bu\|_{Z_{\alpha+1}} \\
&\leq \|C\|_{\mathcal{L}(Z_{\alpha+1}, Y)} \|(sI + A)^{-1}\|_{\mathcal{L}(Z_{\alpha}, Z_{\alpha+1})} \|Bu\|_{Z_{\alpha}} \\
&\leq \|C\|_{\mathcal{L}(Z_{\alpha+1}, Y)} \|(sI + A)^{-1}\|_{\mathcal{L}(Z_{\alpha}, Z_{\alpha+1})} \|B\|_{\mathcal{L}(U, Z_{\alpha})} \|u\|_U.
\end{aligned}$$

Thus for each  $s \in \rho(-A)$ ,  $\Theta(s) \in \mathcal{L}(U, Y)$ . In fact we have shown that

$$\|C(sI + A)^{-1}B\|_{\mathcal{L}(U, Y)} \leq \|C\|_{\mathcal{L}(Z_{\alpha+1}, Y)} \|(sI + A)^{-1}\|_{\mathcal{L}(Z_{\alpha}, Z_{\alpha+1})} \|B\|_{\mathcal{L}(U, Z_{\alpha})}. \quad (2.29)$$

3. Using the resolvent identity, it is easy to check that for  $s$  and  $\beta$  in  $\rho(-A)$ , we have

$$\Theta(s) - \Theta(\beta) = -(s - \beta)C(sI + A)^{-1}(\beta + A)^{-1}B.$$

4. Finally we show that the system  $\Sigma$  given by the triple  $(-A, B, C)$  is well-posed by showing that the map  $\Theta(\cdot)$  is bounded in the half-plane  $\mathbb{C}_0^+ := \{s \in \mathbb{C} \mid \operatorname{Re}(s) > 0\}$ . In order to do this, we use the exponential stability of the generator of the semigroup and (2.29): We have for all  $s \in \mathbb{C}_0^+$

$$\begin{aligned}
\|(sI + A)^{-1}\|_{\mathcal{L}(Z_{\alpha}, Z_{\alpha+1})} &\leq \|A(sI + A)^{-1}\|_{\mathcal{L}(Z_{\alpha})} \\
&= \|I - s(sI + A)^{-1}\|_{\mathcal{L}(Z_{\alpha})} \\
&\leq \|I\|_{\mathcal{L}(Z_{\alpha})} + |s| \|(sI + A)^{-1}\|_{\mathcal{L}(Z_{\alpha})} \\
&\leq 1 + |s| \frac{M}{|s|} \quad (\text{using Theorem 5.2.(c), Pazy [63]}) \\
&= 1 + M,
\end{aligned}$$

for some  $M > 0$ . This completes the proof.  $\blacksquare$

**Remark:** A more general result with unbounded  $C$  can be proven in a similar manner. We quote the following from Staffans [77]:

**Theorem 2.4.4** *Let  $-A$  generate an exponentially stable analytic strongly continuous semigroup  $\{T(t)\}_{t \geq 0}$  in  $Z$ ,  $B \in \mathcal{L}(U, Z_{\alpha_B})$ ,  $C \in \mathcal{L}(Z_{\alpha_C}, Y)$  and  $D \in \mathcal{L}(U, Y)$ , where  $\alpha_B \leq \alpha_C < \alpha_B + 1$ . Fix any  $\gamma$  satisfying  $\alpha_C - \frac{1}{2} < \gamma < \alpha_B + \frac{1}{2}$ . Then  $-A$ ,  $B$ ,  $C$  generate a regular well-posed system on  $(U, Z_{\gamma}, Y)$ , where  $X = Z_{\gamma}$ . The transfer function is given by  $G(s) = C(sI + A)^{-1}B + D$ , and it satisfies*

$$\lim_{\substack{|s| \rightarrow \infty \\ s \in \mathbb{C}_0^+}} G(s) = D.$$

In the sequel, the operators  $L_B L_C$  and  $L_C L_B$  play an important role, so we collect some properties here which readily follow from Lemmas 2.4.1.2 and 2.4.2.2.

**Corollary 2.4.5**  $L_B$  and  $L_C$  defined by (2.16) and (2.24), respectively, satisfy:

1.  $L_B L_C \in \mathcal{L}(Z_\beta)$  for any  $\beta$  satisfying  $-1 < \beta < 1 + \alpha$ .
2.  $L_C L_B \in \mathcal{L}(Z'_\gamma)$  for any  $\gamma$  satisfying  $-(1 + \alpha) < \gamma < 1$ .

**Proof** From part 2 of Lemma 2.4.1, we know that  $L_B \in \mathcal{L}(Z'_\gamma, Z_\beta)$  for  $\gamma, \beta$  satisfying  $\beta - \gamma < 1 + 2\alpha, \beta < 1 + \alpha, \gamma > -(1 + \alpha)$ . Furthermore, part 2 of Lemma 2.4.2 states that  $L_C \in \mathcal{L}(Z_\beta, Z'_\gamma)$  for  $\gamma, \beta$  satisfying  $\gamma - \beta < 1, \beta > -1, \gamma < 1$ .

Since  $-1 < \alpha$ , the intersection of the feasible regions depicted in Figures 2.4 and 2.5 is nonempty and is sketched in Figure 2.6.

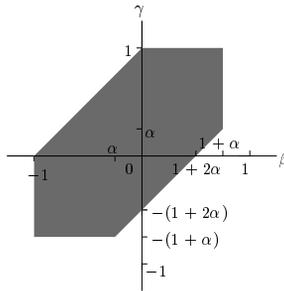


Figure 2.6: The intersection of feasible regions.

Hence the claims in parts 1 and 2 follow. ■

We now examine the Hankel operator associated with  $h(\cdot) = CT(\cdot)B$ . Throughout the remainder of this section, we assume that the input and output spaces are finite-dimensional: say,  $U = \mathbb{C}^m$  and  $Y = \mathbb{C}^p$ .

**Lemma 2.4.6** *If  $A, B,$  and  $C$  satisfy the assumptions A1, A2 and A3 listed at the beginning of this section with  $U = \mathbb{C}^m$  and  $Y = \mathbb{C}^p$ , then:*

1. *The impulse response  $h(\cdot) = CT(\cdot)B \in L_1([0, \infty), \mathbb{C}^{p \times m})$  and so it has a well-defined compact Hankel operator  $\Gamma : L_2([0, \infty), \mathbb{C}^m) \rightarrow L_2([0, \infty), \mathbb{C}^p)$  given by*

$$(\Gamma u)(t) = \int_0^\infty h(t + \tau)u(\tau)d\tau \quad t \geq 0, \quad \text{for } u \in L_2([0, \infty), \mathbb{C}^m). \quad (2.30)$$

2. *If  $-\frac{1}{2} < \beta < \frac{1}{2} + \alpha$ , then  $\Gamma = \mathcal{C}\mathcal{B}$ , where  $\mathcal{B} \in \mathcal{L}(L_2([0, \infty), \mathbb{C}^m), Z_\beta)$  and  $\mathcal{C} \in \mathcal{L}(Z_\beta, L_2([0, \infty), \mathbb{C}^p))$ .*

**Proof 1.** We have

$$\begin{aligned}
\|CT(t)Bu\|_{\mathbb{C}} &\leq \|C\|_{\mathcal{L}(Z, \mathbb{C})} \|T(t)Bu\|_Z \\
&\leq \|C\|_{\mathcal{L}(Z, \mathbb{C})} \|A^{-\alpha}T(t)A^\alpha Bu\|_Z \\
&\leq \|C\|_{\mathcal{L}(Z, \mathbb{C})} \|A^{-\alpha}T(t)\|_{\mathcal{L}(Z)} \|A^\alpha B\|_{\mathcal{L}(\mathbb{C}, Z)} \|u\|_{\mathbb{C}} \\
&\leq \|C\|_{\mathcal{L}(Z, \mathbb{C})} \frac{Ke^{-\epsilon t}}{t^{-\alpha}} \|A^\alpha B\|_{\mathcal{L}(\mathbb{C}, Z)} \|u\|_{\mathbb{C}},
\end{aligned}$$

and since  $-\alpha < 1$ , it follows that  $h(\cdot) \in L_1([0, \infty), \mathbb{C}^{\times n})$ . Compactness follows from Lemma 8.2.4 (page 399, Curtain and Zwart [27]).

2. From part 1 of Lemma 2.4.1 it follows that  $\mathcal{B} \in \mathcal{L}(L_2([0, \infty), \mathbb{C}^m), Z_\beta)$  and from part 1 of Lemma 2.4.2,  $\mathcal{C} \in \mathcal{L}(Z_\beta, L_2([0, \infty), \mathbb{C}))$ . Consequently, if  $u \in L_2([0, \infty), \mathbb{C}^m)$ ,  $Bu \in Z_\beta$ , and

$$(\mathcal{C}(Bu))(t) = CT(t)Bu = CT(t) \int_0^\infty T(\tau)Bu(\tau)d\tau.$$

But  $CT(t) \in \mathcal{L}(Z_\beta, \mathbb{C})$  and so we have (see for example Theorem A.5.23, page 628, Curtain and Zwart [27])

$$\begin{aligned}
(\mathcal{C}(Bu))(t) &= \int_0^\infty CT(t)T(\tau)Bu(\tau)d\tau = \int_0^\infty CT(t+\tau)Bu(\tau)d\tau \\
&= \int_0^\infty h(t+\tau)u(\tau)d\tau = (\Gamma u)(t).
\end{aligned}$$

■

**Remark:** In fact,  $\Gamma$  is nuclear, that is  $\sum_{k=1}^\infty \sigma_k < \infty$  (see Theorem 3.2.3 in Chapter 3).

We will first show that  $L_B L_C \in \mathcal{L}(Z_\beta)$  is compact for all  $\beta$  satisfying  $-1 < \beta < 1 + \alpha$ . This was shown in Curtain and Ichikawa [14] under the extra assumption that  $T(t)$  was compact for each  $t > 0$ . Here we do not need this assumption, since we assume that  $B$  has finite rank.

**Lemma 2.4.7** *Under the assumptions A1, A2 and A3, with  $U = \mathbb{C}^m$  and  $Y = \mathbb{C}$ ,  $L_B L_C \in \mathcal{L}(Z_\beta)$  is compact for all  $\beta$  satisfying  $-1 < \beta < 1 + \alpha$ .*

**Proof** The map  $t \mapsto T(t)BB'T(t)' : (0, \infty) \rightarrow \mathcal{L}(Z'_\gamma, Z_\beta)$  is continuous in the norm topology, and since  $B$  is compact, for each  $t$  greater than zero  $T(t)BB'T(t)'$  is compact. Let  $\mathcal{C}(Z'_\gamma, Z_\beta)$  denote the Banach space of compact operators from  $Z'_\gamma$  to  $Z_\beta$ . Thus we have that the map  $t \mapsto T(t)BB'T(t)' : (0, \infty) \rightarrow \mathcal{C}(Z'_\gamma, Z_\beta)$  is continuous in the norm topology. Moreover, as in the

proof of part 3 of Lemma 2.4.1, it can be shown that for  $\gamma$  and  $\beta$  satisfying  $\beta - \gamma < 1 + 2\alpha$ ,  $\beta < 1 + \alpha$  and  $\gamma > -(1 + \alpha)$ , we have

$$\int_0^\infty \|T(t)BB'T(t')\|_{\mathcal{C}(Z'_\gamma, Z_\beta)} dt < \infty.$$

Consequently, the Bochner integral  $\int_0^\infty T(t)BB'T(t')dt \in \mathcal{C}(Z'_\gamma, Z_\beta)$  (see for example, Thomas [81] or Hille and Phillips [46]). Proceeding as in the proof of Corollary 2.4.5.1, it follows that  $L_B L_C \in \mathcal{L}(Z_\beta)$  and is compact for all  $\beta$  satisfying  $-1 < \beta < 1 + \alpha$ . ■

Next we show that the spectra of  $\Gamma^* \Gamma \in \mathcal{L}(L_2([0, \infty), \mathbb{C}^m))$  and  $L_B L_C \in \mathcal{L}(Z_\beta)$  are identical for all  $\beta$  satisfying  $-1 < \beta < 1 + \alpha$ .

**Lemma 2.4.8** *Under the assumptions A1, A2 and A3 with  $U = \mathbb{C}^m$  and  $Y = \mathbb{C}^p$ , the nonzero Hankel singular values are equal to the square roots of the nonzero eigenvalues of  $L_B L_C \in \mathcal{L}(Z_\beta)$  for  $\beta$  satisfying  $-1 < \beta < 1 + \alpha$ .*

**Proof** First we prove the result for  $-\frac{1}{2} < \beta < \frac{1}{2} + \alpha$ . From part 3 of Lemma 2.4.1, we know that  $L_B = \mathcal{B}\mathcal{B}' \in \mathcal{L}(Z'_\gamma, Z_\beta)$  for  $\beta < \frac{1}{2} + \alpha$  and  $\gamma > -\frac{1}{2} - \alpha$ . Furthermore, from part 3 of Lemma 2.4.2,  $L_C = \mathcal{C}'\mathcal{C} \in \mathcal{L}(Z_\beta, Z'_\gamma)$  for  $\beta > -\frac{1}{2}$  and  $\gamma < \frac{1}{2}$ .

Let us choose  $\beta$  and  $\gamma$  satisfying

$$-\frac{1}{2} - \alpha < \gamma < \frac{1}{2}, \quad -\frac{1}{2} < \beta < \frac{1}{2} + \alpha.$$

(We remark that such a choice is possible, since  $\alpha > -1$ .) Consequently,

$$L_B L_C = \mathcal{B}\mathcal{B}'\mathcal{C}'\mathcal{C} = \mathcal{B}\Gamma^*\mathcal{C} \in \mathcal{L}(Z_\beta),$$

and owing to the compactness of  $\Gamma$  (and hence of  $\Gamma^*$ ),  $L_B L_C \in \mathcal{L}(Z_\beta)$  is compact for  $-\frac{1}{2} < \beta < \frac{1}{2} + \alpha$ .

Let  $-\frac{1}{2} < \beta < \frac{1}{2} + \alpha$  and suppose that  $\lambda \neq 0$  is an eigenvalue of  $L_B L_C \in \mathcal{L}(Z_\beta)$  and  $v$  a corresponding eigenvector;  $L_B L_C v = \lambda v$ . Then

$$\Gamma\Gamma^*\mathcal{C}v = \mathcal{C}\mathcal{B}\mathcal{B}'\mathcal{C}'\mathcal{C}v = \mathcal{C}L_B L_C v = \mathcal{C}(\lambda v) = \lambda\mathcal{C}v,$$

and so  $\lambda$  is an eigenvalue of  $\Gamma\Gamma^*$ , since  $\mathcal{C}v \neq 0$  (for otherwise  $L_B L_C v = L_B \mathcal{C}'\mathcal{C}v = 0$ , a contradiction).

Conversely, suppose that  $\mu$  is an eigenvalue of  $\Gamma\Gamma^*$ , with corresponding eigenvector  $y$ . Then

$$L_B L_C L_B \mathcal{C}'y = L_B L_C \mathcal{B}\mathcal{B}'\mathcal{C}'y = L_B L_C \mathcal{B}\Gamma^*y,$$

and

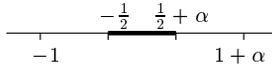
$$L_B L_C L_B C' y = B B' C' C B B' C' y = B \Gamma^* \Gamma^* y = B \Gamma^* (\mu y) = \mu B \Gamma^* y.$$

So  $\mu$  is an eigenvalue of  $L_B L_C$ , since  $B \Gamma^* y \neq 0$  (for otherwise  $\Gamma^* y = C B \Gamma^* y = 0$ , a contradiction).

Finally, we extend the result to all  $\beta$  satisfying  $-1 < \beta < 1 + \alpha$ . Since  $L_B L_C \in \mathcal{L}(Z_\beta)$  is compact for  $-1 < \beta < 1 + \alpha$ , it has a pure point spectrum. Moreover, by Lemmas 2.4.1 and 2.4.2,

$$L_B L_C \in \mathcal{L}(Z_\beta, Z_{\beta+\theta}) \text{ for } -1 < \beta < 1 + \alpha - \theta,$$

for any  $\theta$  satisfying  $0 < \theta < \min\{1 + \alpha, \frac{1}{2}\}$ . From this it follows that the nonzero point spectra of  $L_B L_C \in \mathcal{L}(Z_\beta)$  and  $L_B L_C \in \mathcal{L}(Z_{\beta+\theta})$  are the same for all  $\beta \in (-1, 1 + \alpha - \theta)$ .



Hence the nonzero point spectrum of  $L_B L_C \in \mathcal{L}(Z_\beta)$  is independent of  $\beta$  and so  $\sigma(L_B L_C) = \sigma(\Gamma^* \Gamma)$  for all  $\beta$  satisfying  $-1 < \beta < 1 + \alpha$ . ■

Let  $\sigma_1 \geq \sigma_2 \geq \dots \geq 0$  denote the Hankel singular values of the system. Furthermore, let  $\sigma$  be such that  $\sigma_{1+1} < \sigma < \sigma_1$ . Then we have the following useful lemma which will be used in the next section.

**Lemma 2.4.9** *Under the assumptions A1, A2 and A3 with  $U = \mathbb{C}^m$  and  $Y = \mathbb{C}$ , if  $\sigma_{1+1} < \sigma < \sigma_1$ , then  $J := I - \frac{1}{\sigma^2} \Gamma^* \Gamma \in \mathcal{L}(L_2([0, \infty), \mathbb{C}^m))$  has a spectrum  $\sigma(J)$  contained in  $(-\infty, -\delta) \cup (\delta, \infty)$  for some  $\delta > 0$ , and  $\sigma(J) \cap (-\infty, -\delta)$  consists of exactly 1 negative eigenvalues.*

**Proof**  $\Gamma^* \Gamma$  is compact and has a pure point spectrum with 0 as the accumulation point. Considering the resolvent  $[(I - \frac{1}{\sigma^2} \Gamma^* \Gamma) - \lambda I]^{-1}$ , it is easy to see that  $J$  has a spectrum which is a shifted version of the spectrum of  $\frac{1}{\sigma^2} \Gamma^* \Gamma$ . Finally, since  $\Gamma^* \Gamma$  has a pure point spectrum  $\{\sigma_1^2, \sigma_2^2, \dots\}$  with 0 as the accumulation point, and since  $\sigma_{1+1} < \sigma < \sigma_1$ ,  $J$  has exactly 1 negative eigenvalues. ■

Analytic semigroups are generated by parabolic and some hyperbolic partial differential equations. For a change, we give a different example of an analytic system from the class of fractional transfer functions.

**Example 2.4.10** *(The fractional transfer function  $\frac{1}{(1+s)^m}$ , where  $0 < m < 1$ .)*

Let  $\alpha \in \mathbb{R}$  satisfy  $-1 < \alpha < m - 1 (< 0)$ . We denote by  $Z$  the vector space

$$Z = \left\{ f \in L_2(0, \infty) \mid \int_0^\infty (1+x)^{-\alpha} |f(x)|^2 dx < \infty \right\},$$

with the inner product

$$\langle f_1, f_2 \rangle_Z = \int_0^\infty (1+x)^{-\alpha} f_1(x) \overline{f_2(x)} dx.$$

It is easy to see that  $Z$  is a Hilbert space. Let  $R_A : Z \rightarrow Z$  be the multiplication operator by  $x \mapsto \frac{1}{1+x}$ :

$$(R_A f)(x) = \frac{1}{1+x} f(x) \text{ for all } f \in Z.$$

The operator  $R_A \in \mathcal{L}(Z)$ , since

$$\|R_A f\|_Z^2 = \int_0^\infty (1+x)^{-\alpha} \frac{1}{(1+x)^2} |f(x)|^2 dx \leq \int_0^\infty (1+x)^{-\alpha} |f(x)|^2 dx = \|f\|_Z^2.$$

Furthermore,  $R_A$  is injective, since if  $0 = (R_A f)(x) = \frac{1}{1+x} f(x)$  for almost all  $x \in (0, \infty)$ , then  $f(x) = 0$  for almost all  $x \in (0, \infty)$ . Thus one can define an algebraic inverse of  $R_A$ , which we denote by  $A$ , as follows:

$$D(A) = \text{ran}(R_A) \text{ and } AR_A f = f.$$

It is immediate that  $A$  is well-defined. The operator  $A$  is closed, since the graphs of  $A$  and  $R_A$  are related by the following isomorphism:

$$\mathbb{I} : Z \times Z \rightarrow Z \times Z, \quad \mathbb{I}(f_1, f_2) = (f_2, f_1) \text{ for all } (f_1, f_2) \in Z \times Z,$$

We have

$$D(A) = \text{ran}(R_A) = \{R_A f \in Z \mid f \in Z\} = \{g \in Z \mid (1+\cdot)g(\cdot) \in Z\}.$$

Moreover, if  $g \in D(A)$ ,  $(Ag)(x) = (1+x)g(x)$ . The set  $D(A)$  contains continuous functions compactly supported in  $(0, \infty)$ , that is,  $\mathcal{C}_{00}(0, \infty) \subset D(A)$ . If  $f \in Z$ , then  $(1+\cdot)^{-\frac{\alpha}{2}} f(\cdot) \in L_2(0, \infty)$ . Since  $\mathcal{C}_{00}(0, \infty)$  is dense in  $L_2(0, \infty)$ , there exists a sequence  $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{C}_{00}(0, \infty)$  such that  $f_n \rightarrow (1+\cdot)^{-\frac{\alpha}{2}} f(\cdot)$  in  $L_2(0, \infty)$ , that is

$$(1+\cdot)^{-\frac{\alpha}{2}} \frac{f_n(\cdot)}{(1+\cdot)^{-\frac{\alpha}{2}}} \xrightarrow{L_2(0, \infty)} (1+\cdot)^{-\frac{\alpha}{2}} f(\cdot).$$

Hence we have  $\frac{f_n(\cdot)}{(1+\cdot)^{-\frac{\alpha}{2}}} \xrightarrow{Z} f$ , where  $\frac{f_n(\cdot)}{(1+\cdot)^{-\frac{\alpha}{2}}} \in \mathcal{C}_{00}(0, \infty)$ . So  $D(A)$  is dense in  $Z$ .

It is clear that  $R_A = R_A^*$ : indeed, for any  $f$  and  $g$  in  $Z$ , we have

$$\begin{aligned} \langle R_A f, g \rangle_Z &= \int_0^\infty (1+x)^{-\alpha} \frac{1}{1+x} f(x) \overline{g(x)} dx \\ &= \int_0^\infty (1+x)^{-\alpha} f(x) \overline{\left[ \frac{g(x)}{1+x} \right]} dx \\ &= \langle f, R_A g \rangle_Z. \end{aligned}$$

Thus from Lemma A.3.65.(c) (page 603, Curtain and Zwart [27]), we have

$$(A^*)^{-1} = (A^{-1})^* = R_A^* = R_A.$$

So  $R_A A^* f = f$  for all  $f \in D(A^*)$ ,  $A^* R_A f = f$  for all  $f \in Z$  and  $D(A^*) = \text{ran}(R_A) = D(A)$ . Furthermore, if  $f \in D(A) = D(A^*)$ , then

$$A f = A (R_A A^* f) = (A R_A) (A^* f) = A^* f$$

and so  $A$  is self-adjoint. If  $f \in D(A)$ , then we have

$$\begin{aligned} \langle A f, f \rangle_Z &= \int_0^\infty (1+x)^{-\alpha} (1+x) f(x) \overline{f(x)} dx \\ &= \int_0^\infty (1+x)^{-\alpha} (1+x) |f(x)|^2 dx \\ &\geq \int_0^\infty (1+x)^{-\alpha} |f(x)|^2 dx \\ &= \|f\|_Z^2. \end{aligned}$$

$A$  is a nonnegative self-adjoint operator and so using Example 1.25 (page 493, Kato [49]) it follows that  $-A$  is the infinitesimal generator of an analytic semigroup  $\{T(t)\}_{t \geq 0}$ .

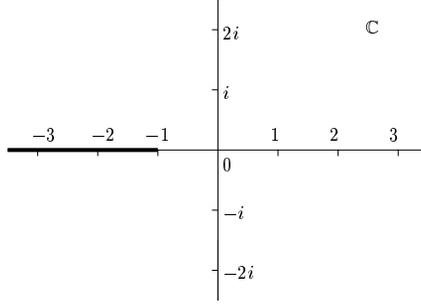
The resolvent of  $-A$ . It can be shown that  $\rho(-A) = \mathbf{C}(-\infty, -1]$  and for  $\lambda \in \rho(-A)$ ,

$$(\lambda I + A)^{-1} f(x) = \frac{1}{\lambda + 1 + x} f(x) \quad \text{for all } f \in Z.$$

The point spectrum of  $-A$  is empty.  $\sigma(-A) = \mathbf{C}(\mathbf{C}(-\infty, -1]) = (-\infty, -1]$ . Let  $-A f = \lambda f$ ,  $f \in D(A)$ , and  $f \neq 0$ . So  $(-A f)(x) = \lambda f(x)$ , that is,  $-(1+x)f(x) = \lambda f(x)$ . But since  $x \neq \lambda + 1$  almost everywhere,  $f(x) = 0$  almost everywhere. Thus  $f = 0$  (a contradiction!). So the point spectrum of  $A$  is empty, and its (continuous) spectrum is  $(-\infty, -1]$ , see Figure 2.7.

The semigroup  $\{T(t)\}_{t \geq 0}$ . Given  $t \geq 0$ , define  $T_*(t) \in \mathcal{L}(Z)$  by  $(T_*(t)f)(x) = e^{-(1+x)t} f(x)$ . The operator  $T_*(t)$  is well-defined since for all  $f \in Z$ , we have

$$\|T_*(t)f\|_Z^2 = \int_0^\infty (1+x)^{-\alpha} |(T_*(t)f)(x)|^2 dx$$

Figure 2.7:  $\sigma(A) = \sigma_c(A) = (-\infty, -1]$ .

$$\begin{aligned}
 &= \int_0^\infty (1+x)^{-\alpha} e^{-2(1+x)t} |f(x)|^2 dx \\
 &\leq e^{-2t} \int_0^\infty (1+x)^{-\alpha} |f(x)|^2 dx \\
 &= e^{-2t} \|f\|_Z^2.
 \end{aligned} \tag{2.31}$$

Next we prove that  $\{T_*(t)\}_{t \geq 0}$  is a strongly continuous semigroup on  $Z$ :

1.  $(T_*(0)f)(x) = f(x)$  for all  $f \in Z$ , and so  $T_*(0) = I$ .
2. For any  $f \in Z$ , we have

$$\begin{aligned}
 (T_*(\tau+t)f)(x) &= e^{-(1+x)(\tau+t)} f(x) \\
 &= e^{-(1+x)\tau} \left[ e^{-(1+x)t} f(x) \right] \\
 &= e^{-(1+x)\tau} (T_*(t)f)(x) \\
 &= (T_*(\tau)(T_*(t)f))(x).
 \end{aligned}$$

Thus  $T_*(\tau+t) = T_*(\tau)T(t)$  for all nonnegative  $t$  and  $\tau$ .

3. Finally we prove the strong continuity at  $t = 0$ . Let  $0 \neq f \in Z$ . We have

$$\|T_*(t)f - f\|_Z^2 = \int_0^\infty (1+x)^{-\alpha} \left| e^{-(1+x)t} - 1 \right|^2 |f(x)|^2 dx.$$

Choose a  $M > 0$  large enough such that  $0 < \int_M^\infty |f(x)|^2 dx < \frac{\epsilon}{2}$ . Next, choose a  $\delta > 0$  small enough so that  $0 \leq t < \delta$  implies that

$$\sup_{x \in [0, M]} \left| e^{-(1+x)t} - 1 \right| < \frac{\epsilon}{2 \int_0^M (1+x)^{-\alpha} |f(x)|^2 dx}.$$

So  $\lim_{t \searrow 0} \|T_*(t)f - f\|_2 = 0$ , for all  $f \in Z$ . This proves the strong continuity.

Hence  $\{T_*(t)\}_{t \geq 0}$  is a strongly continuous semigroup on  $L_2(0, \infty)$ .

Next we prove that the generator of  $\{T_*(t)\}_{t \geq 0}$  is  $(-A, D(A))$ . Let us denote the infinitesimal generator of  $\{T_*(t)\}_{t \geq 0}$  by  $(A_*, D(A_*))$ . From Lemma 2.1.11 (page 24, Curtain and Zwart [27]), for all  $f \in Z$  we have

$$(\lambda I - A_*)^{-1} f = \int_0^\infty e^{-\lambda t} T_*(t) f dt$$

for  $\text{Re}(\lambda) > \omega_{0*}$ , the growth bound of the semigroup  $\{T_*(t)\}_{t \geq 0}$ . Consequently, for  $\lambda \in \mathbb{C}$  such that  $\text{Re}(\lambda) > \max\{-1, \omega_{0*}\}$  we obtain

$$\begin{aligned} ((\lambda I - A_*)^{-1} f)(x) &= \int_0^\infty e^{-\lambda t} (T_*(t) f)(x) dt \\ &= \frac{1}{\lambda + 1 + x} f(x) \\ &= ((\lambda I + A)^{-1} f)(x). \end{aligned}$$

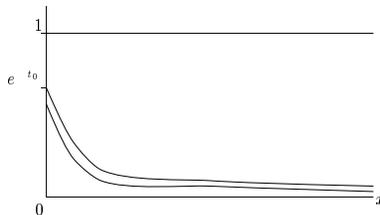
From this it follows easily that  $D(A) = D(A_*)$  and  $A_* = -A$ .

Since the semigroup is determined by its infinitesimal generator (see for example, Theorem 5.5, page 21, Pazy [63]), it follows that the semigroup  $\{T(t)\}_{t \geq 0}$  is in fact the semigroup  $\{T_*(t)\}_{t \geq 0}$ .

From (2.31) it follows that  $\|T(t)\| \leq e^{-t}$  and so the growth bound of  $\{T(t)\}_{t \geq 0}$  is less than  $-1$ . Hence the semigroup  $\{T(t)\}_{t \geq 0}$  is exponentially stable.

**Remarks:**

1. Since the semigroup  $\{T(t)\}_{t \geq 0}$  is analytic, the map  $T(\cdot) : (0, \infty) \rightarrow \mathcal{L}(Z)$  is continuous in the  $\mathcal{L}(Z)$ -topology (see for example, Proposition 2.1.1.(iv), page 35, Lunardi [53]). This can also be seen directly:



First we show that for  $\alpha, \beta$  both greater than zero,

$$\sup_{x \in (0, \infty)} \left| e^{-\alpha(1+x)} - e^{-\beta(1+x)} \right| < \frac{|\alpha - \beta|}{\min\{\alpha, \beta\}}.$$

From the mean value theorem applied to the function  $\theta \mapsto e^{-\theta(1+x)} : (0, \infty) \rightarrow \mathbb{R}$ , we have

$$\frac{e^{-\alpha(1+x)} - e^{-\beta(1+x)}}{\alpha - \beta} = (x+1)e^{-\gamma(1+x)},$$

for some  $\gamma \in (\min\{\alpha, \beta\}, \max\{\alpha, \beta\})$ . It is easy to see that the function  $x \mapsto (x+1)e^{-\gamma(1+x)} : (0, \infty) \rightarrow \mathbb{R}$  is bounded by  $\frac{1}{\gamma}$ : we have

$$e^{\gamma(1+x)} = 1 + \gamma(1+x) + \frac{[\gamma(1+x)]^2}{2!} + \dots > \gamma(1+x),$$

and so  $(1+x)e^{-\gamma(1+x)} < \frac{1}{\gamma}$ . Finally, using the fact that  $\gamma > \min\{\alpha, \beta\}$ , we have the desired result.

Thus for every  $t_0 > 0$ ,  $\lim_{t \rightarrow 0} \|T(t+t_0) - T(t_0)\| = 0$ .

2. However, the semigroup  $\{T(t)\}_{t \geq 0}$  is not continuous at  $t = 0$  in the norm topology. Indeed, if it were, then its infinitesimal generator  $A$  would be bounded (see for example Theorem 1.2, page 2, Pazy [63]); but we know that  $A$  is unbounded, since  $D(A)$  is not closed.

We note that  $A^\theta h(x) = (1+x)^\theta h(x)$ ,  $h \in D(A^\theta)$ , and for each  $\theta \in \mathbb{R}$ , we define  $Z_\theta$  as described earlier.

Define  $B : \mathbb{C} \rightarrow Z_\alpha$  by  $(Bu)(x) = x^{-\frac{m}{2}}u$ . Using

$$\int_0^\infty x^p(1+x)^q dx = \frac{\Gamma(p)\Gamma(1-p-q)}{\Gamma(-q)} \quad \text{if } p > -1 \text{ and } p+q < -1, \quad (2.32)$$

it is easy to see that  $B \in \mathcal{L}(\mathbb{C}, Z_\alpha)$ .

Let  $C : Z \rightarrow \mathbb{C}$  be given by

$$Cf = \frac{\sin(m\pi)}{\pi} \int_0^\infty x^{-\frac{m}{2}} f(x) dx \quad \text{for all } f \in Z.$$

From (2.32) and the Cauchy-Schwarz inequality it follows that  $C \in \mathcal{L}(Z, \mathbb{C})$ .

Thus all the conditions in Theorem 2.4.3 are satisfied and so the triple  $(-A, B, C)$  defines a regular well-posed linear system on  $Z_\beta$ , where  $-\frac{1}{2} < \beta < \frac{1}{2} + \alpha$ , and it realizes the transfer function  $G(s) = \frac{1}{(1+s)^m}$ : Indeed we have

$$\begin{aligned} G(s) &= C(sI + A)^{-1}B \\ &= \frac{\sin(m\pi)}{\pi} \int_0^\infty x^{-\frac{m}{2}} (s+1+x)^{-1} x^{-\frac{m}{2}} dx \\ &= \frac{\sin(m\pi)}{\pi} \int_0^\infty \frac{x^{-m}}{s+1+x} dx \\ &= \frac{1}{(1+s)^m}, \end{aligned}$$

where we choose the following branch: for  $s \in \mathbb{C} \setminus (-\infty, -1]$ ,

$$\frac{1}{(1+s)^m} = e^{-m[\log_{1+s} + i \operatorname{Arg}(1+s)]}, \quad -\pi < \operatorname{Arg}(1+s) < \pi$$

(see for instance pages 187-188, Lang [51]).

◇

## 2.5 Transfer function algebras

In this section, we introduce some function spaces which will be used in the sequel. Throughout this book, we use the following notation. For  $r \in \mathbb{R}$  (see Figure 2.8)

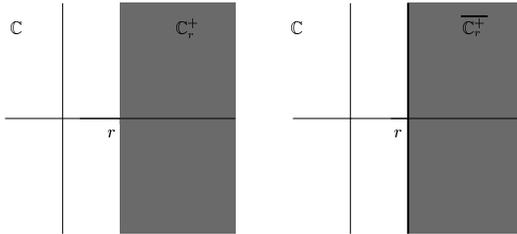


Figure 2.8: The half-planes  $\mathbb{C}_r^+$  and  $\overline{\mathbb{C}_r^+}$ .

$$\begin{aligned} \mathbb{C}_r^+ &= \{s \in \mathbb{C} \mid \operatorname{Re}(s) > r\}, & \overline{\mathbb{C}_r^+} &= \{s \in \mathbb{C} \mid \operatorname{Re}(s) \geq r\}, \\ \mathbb{C}_r^- &= \{s \in \mathbb{C} \mid \operatorname{Re}(s) < r\}, & \overline{\mathbb{C}_r^-} &= \{s \in \mathbb{C} \mid \operatorname{Re}(s) \leq r\}. \end{aligned}$$

1.  $H_\infty^c(\mathbb{C}^{p \times m})$  denotes the set of complex  $p \times m$  matrix-valued functions defined in the closed right half-plane, which are bounded and analytic in  $\mathbb{C}_0^+$ , and continuous in  $\overline{\mathbb{C}_0^+}$ . The set  $H_\infty^c(\mathbb{C})$ , with point-wise addition and multiplication, is a commutative ring with identity. Note that if  $G \in H_\infty^c(\mathbb{C}^{p \times m})$ , then it does not necessarily follow that it has a limit at infinity. So for instance,  $e^{-s}$  is an element belonging to  $H_\infty^c(\mathbb{C})$ .
2.  $H_{\infty,1}^c(\mathbb{C}^{p \times m})$  denotes the set of complex  $p \times m$  matrix-valued functions  $X(\cdot)$  of a complex variable defined in the closed right half-plane with a decomposition  $X = G_* + F$ , where  $G_*$  is the matrix transfer function of a system of MacMillan degree *at most equal* to 1, with all its poles in the open right half-plane, and  $F \in H_\infty^c(\mathbb{C}^{p \times m})$ .
3.  $H_{\infty,[1]}^c(\mathbb{C}^{p \times m})$  denotes the set of complex  $p \times m$  matrix-valued functions  $X(\cdot)$  of a complex variable defined in the closed right half-plane with a decomposition  $X = G_* + F$ , where  $G_*$  is the matrix transfer function of

a system of MacMillan degree *equal* to 1, with all its 1 poles in the open right half-plane, and  $F \in H_\infty^c(\mathbb{C}^{\times m})$ .

4.  $\mathcal{S}_\infty$  denotes the set of complex-valued functions  $g \in H_\infty^c(\mathbb{C})$  that have a nonzero limit at infinity in  $\overline{\mathbb{C}_0^+}$ , finitely many zeros in  $\overline{\mathbb{C}_0^+}$ , and the zeros are all contained in the open right half-plane.
5.  $\mathcal{R}$  denotes the class of proper, rational functions  $g$  with complex coefficients such that  $g$  has no poles in  $\overline{\mathbb{C}_0^+}$ .
6.  $\mathcal{R}_\infty$  denotes the class of proper, rational functions  $g$  with complex coefficients such that  $g$  has no poles in  $\overline{\mathbb{C}_0^+}$ , and has a nonzero limit at infinity.
7.  $\mathcal{MH}_\infty^c$  denotes the set of matrices (of any size) with each entry in  $H_\infty^c(\mathbb{C})$ . Similarly, we use  $\mathcal{MR}$ ,  $\mathcal{MH}_{\infty, \bullet}^c$ , and so on.

Suppose  $M$  and  $N$  belong to  $\mathcal{MH}_\infty^c$  and have the same number of columns. Then the pair  $(M, N)$  is *right coprime* over  $\mathcal{MH}_\infty^c$  if there exist  $X$  and  $Y$  in  $\mathcal{MH}_\infty^c$  such that the following *Bezout identity* holds:

$$XM - YN = I \text{ for all } s \in \overline{\mathbb{C}_0^+}.$$

Suppose that  $G$  belongs to  $H_{\infty, [1]}^c(\mathbb{C}^{\times m})$  and that the pair  $(M, N)$  is right coprime over  $\mathcal{MH}_\infty^c$ . If  $M$  is such that  $\det(M) \in \mathcal{S}_\infty$  and  $G = NM^{-1}$ , then we call this a *right coprime factorization* of  $G$  over  $\mathcal{MH}_\infty^c$ .

Next we prove a few elementary facts concerning elements from the above classes of transfer functions, which are used in the proofs in the sequel.

**Lemma 2.5.1** *If  $f \in H_\infty^c(\mathbb{C})$  and  $g \in \mathcal{S}_\infty$  has at most 1 zeros, then  $\frac{f}{g} \in H_{\infty, 1}^c(\mathbb{C})$ .*

**Proof**  $\frac{f}{g}$  has a Laurent expansion around each zero  $p_i$  of  $g$ . So

$$\frac{f(s)}{g(s)} = \sum_{i=1}^r \sum_{j=1}^{m_i} \frac{a_{i,j}}{(s-p_i)^j} + \alpha(s),$$

where  $p_1, \dots, p_r$  are the zeros of  $g$  with multiplicities  $m_1, \dots, m_r$  respectively ( $\sum_{i=1}^r m_i \leq 1$ ), and  $\alpha$  is analytic in  $\overline{\mathbb{C}_0^+}$ .

1. Since  $g$  has a nonzero limit at infinity in  $\overline{\mathbb{C}_0^+}$ , given  $\epsilon > 0$ , there exists an  $M$  such that  $|s| > M$  implies that

$$\left| g(s) - \lim_{\substack{s \rightarrow \infty \\ s \in \overline{\mathbb{C}_0^+}}} g(s) \right| < \epsilon.$$

Thus  $\frac{1}{g(\cdot)}$  is bounded for  $|s| > M$ . Since  $f \in H_\infty^c$ ,  $|f(\cdot)|$  is bounded for  $|s| > M$ . Moreover, choosing  $M$  large enough to include all the zeros  $p_i$ 's in the open disk  $\{|s| < M\}$ , we have that  $\left| \sum_{i=1}^r \sum_{j=1}^{m_i} \frac{a_{i,j}}{(-p_i)^j} \right|$  is bounded for  $|s| > M$ . Thus  $\alpha$  is bounded for  $\Xi_0 := \{s \in \mathbb{C}_0^+ \mid |s| > M\}$ .

- Let  $\epsilon := \min\{\operatorname{Re}(p_1), \dots, \operatorname{Re}(p_r)\}$ . Since  $\alpha$  is analytic in  $\mathbb{C}_0^+$ , in particular it is continuous in the compact set  $\Xi_1 := \{s \in \mathbb{C}_0^+ \mid |s| \leq M, \operatorname{Re}(s) \geq \frac{\epsilon}{2}\}$  and  $|\alpha(\cdot)|$  is bounded for  $s \in \Xi_1$ .

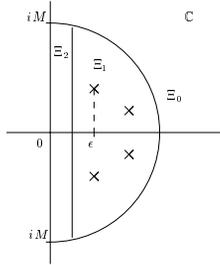


Figure 2.9: The sets  $\Xi_0, \Xi_1$  and  $\Xi_2$ .

- Finally consider the set  $\Xi_2 := \{s \in \mathbb{C}_0^+ \mid |s| < M, \operatorname{Re}(s) < \frac{\epsilon}{2}\}$  (see Figure 2.9). We prove that  $\inf_{s \in \Xi_2} |g(s)| > 0$ . If not, there exists a sequence  $\{s_n\}$  in  $\Xi_2$  such that  $g(s_n) \rightarrow 0$ . Consider the compact set  $\overline{\Xi_2}$ , the closure of  $\Xi_2$ . There exists a convergent subsequence  $\{s_{n_k}\}$  of  $\{s_n\}$  such that  $s_{n_k} \rightarrow s_0 \in \overline{\Xi_2}$  and  $g(s_{n_k}) \rightarrow 0$ . But since  $g \in H_\infty^c(\mathbb{C})$ , it follows that  $g(s_0) = 0$ , which is a contradiction, since the zeros of  $g$  are contained in the complement of  $\overline{\Xi_2}$ . Thus  $\inf_{s \in \Xi_2} |g(s)| > 0$ , which implies that  $\sup_{s \in \Xi_2} \frac{1}{|g(s)|} < \infty$ . Hence  $\left| \frac{f(s)}{g(s)} \right|$  is bounded in  $\Xi_2$ , and since  $\{p_1, \dots, p_r\} \subset \overline{\Xi_2}$ ,  $\left| \sum_{i=1}^r \sum_{j=1}^{m_i} \frac{a_{i,j}}{(-p_i)^j} \right|$  is bounded in  $\Xi_2$ . Thus  $|\alpha(\cdot)|$  is bounded in  $\Xi_2$ .

Since  $\mathbb{C}_0^+ = \Xi_0 \cup \Xi_1 \cup \Xi_2$ , it follows from the above that  $\alpha(\cdot)$  is bounded in  $\mathbb{C}_0^+$ .

Finally, since  $f$  and  $g$  are in  $H_\infty^c$ , and the zeros of  $g$ , namely  $p_1, \dots, p_r$  are contained in  $\mathbb{C}_0^+$ , it follows that

$$s \mapsto \alpha(s) = \frac{f(s)}{g(s)} - \sum_{i=1}^r \sum_{j=1}^{m_i} \frac{a_{i,j}}{(s - p_i)^j}$$

is continuous in  $\overline{\mathbb{C}_0^+} \cap \left\{s \in \overline{\mathbb{C}_0^+} \mid \operatorname{Re}(s) \leq \frac{\epsilon}{2}\right\}$ . Moreover,  $\alpha$  is analytic in  $\mathbb{C}_0^+$ .

Thus  $\alpha$  is continuous in  $\overline{\mathbb{C}_0^+}$  and so  $\alpha \in H_\infty^c(\mathbb{C})$ . This completes the proof of this lemma. ■

**Lemma 2.5.2** *If  $K \in H_{\infty, [1]}^c(\mathbb{C}^{\mathbb{P} \times \mathbb{m}})$ , then there exists a right coprime factorization  $K = NM^{-1}$ , where  $M$  is rational,  $\det(M) \in \mathcal{R}_{\infty}$  has exactly 1 zeros in  $\overline{\mathbb{C}_0^+}$  and they are all contained in  $\mathbb{C}_0^+$ .*

**Proof** Let  $K = G_* + F$ , where  $G_*$  is the matrix transfer function of a system of MacMillan degree equal to  $l$ , with all its 1 poles in the open right half-plane, and  $F \in H_{\infty}^c(\mathbb{C}^{\mathbb{P} \times \mathbb{m}})$ . Let  $G_* = N_1 M^{-1}$  be a right coprime factorization of  $G_*$  over  $\mathcal{MR}$  (which exists by Lemma A.7.37, Curtain and Zwart [27]). We claim that  $K = (FM + N_1)M^{-1}$  is a right coprime factorization over  $\overline{\mathcal{MH}_{\infty}^c}$  for  $K$ .  $M$  is square with  $\det(M) \in \mathcal{R}_{\infty} \subset \mathcal{S}_{\infty}$  having exactly 1 zeros in  $\overline{\mathbb{C}_0^+}$  (this follows for example, from page 287, Rugh [71] and the fact that  $G_*$  has MacMillan degree 1), and they are all contained in  $\mathbb{C}_0^+$  and  $N := FM + N_1 \in \mathcal{MH}_{\infty}^c$  by the ring properties of  $H_{\infty}^c(\mathbb{C})$ . Since  $(N_1, M)$  is a rational coprime pair over  $\mathcal{MR}$ , there exist  $X_*, Y_* \in \mathcal{MR}$  such that  $X_*M - Y_*N_1 = I$ . Defining  $X$  and  $Y$  in  $\mathcal{MH}_{\infty}^c$  by  $X = X_* + Y_*F$ ,  $Y = Y_*$ , we see that

$$XM - YN = [X_* + Y_*F]M - Y_*[FM + N_1] = I$$

and so  $(N, M)$  is a right coprime pair over  $\mathcal{MH}_{\infty}^c$ . ■

**Lemma 2.5.3** *If  $(N, M)$  is a right coprime factorization of  $K \in H_{\infty, [1]}^c(\mathbb{C}^{\mathbb{P} \times \mathbb{m}})$ , and  $V \in H_{\infty}^c(\mathbb{C}^{\mathbb{m} \times \mathbb{m}})$  is invertible as an element of  $\mathcal{MH}_{\infty}^c$ , then  $(NV, MV)$  is also a right coprime factorization of  $K$ .*

*Moreover, any two right coprime factorizations of  $K \in H_{\infty, [1]}^c(\mathbb{C}^{\mathbb{P} \times \mathbb{m}})$  are unique up to a common right multiplication by an invertible element in  $\mathcal{MH}_{\infty}^c$ .*

**Proof** These proofs are analogous to parts *b* and *c* of the proof of Theorem 7.2.8, pages 353-354, Curtain and Zwart [27]. ■

**Lemma 2.5.4** *If*

1.  $M$  and  $N$  belong to  $\mathcal{MH}_{\infty}^c$ ,
2.  $M$  and  $N$  have the same number of columns,
3.  $M$  is a square matrix with  $\det(M) \in \mathcal{S}_{\infty}$  and
4.  $\det(M)$  has 1 zeros in the open right half-plane,

*then  $NM^{-1} \in H_{\infty, 1}^c(\mathbb{C}^{\mathbb{P} \times \mathbb{m}})$ .*

**Proof 1.** The inverse of the square matrix  $M$  is given by

$$M^{-1} = \frac{1}{\det(M)} \operatorname{adj}(M),$$

where  $\operatorname{adj}(M) \in \mathcal{MH}_\infty^c$ , since its components are sums and products of the components of  $M$ , which are all in  $H_\infty^c(\mathbb{C})$  by assumption. Thus from Lemma 2.5.1, we have that all the entries in  $NM^{-1}$  belong to  $H_{\infty,1}^c(\mathbb{C})$  and so  $NM^{-1}$  can be written as the sum of a proper rational matrix  $G_*$  and a matrix  $F \in \mathcal{MH}_\infty^c$ . Thus  $NM^{-1} \in H_{\infty,k}^c(\mathbb{C}^{\times m})$ , for some nonnegative integer  $k$ . In the remainder of the proof, we will show that  $k \leq 1$ , hence proving the claim.

2. Since  $NM^{-1} \in H_{\infty,k}^c(\mathbb{C}^{\times m})$ , it follows from Lemma 2.5.2 that it has a right coprime factorization  $N_*M_*^{-1}$ , where  $M_*$  is rational,  $\det(M_*) \in \mathcal{R}_\infty$  has exactly  $k$  zeros in  $\overline{\mathbb{C}_0^+}$  and they are all contained in  $\mathbb{C}_0^+$ . Since  $(M_*, N_*)$  is right coprime, there exist  $X$  and  $Y$  in  $\mathcal{MH}_\infty^c$  such that

$$XM_* - YN_* = I.$$

Upon post multiplication by  $M_*^{-1}M$ , we obtain

$$XM - YN_*M_*^{-1}M = M_*^{-1}M.$$

Using  $N_*M_*^{-1} = NM^{-1}$ , we have  $M_*^{-1}M = XM - YN \in \mathcal{MH}_\infty^c$ . Consequently  $\frac{\det(M)}{\det(M_*)} \in H_\infty^c(\mathbb{C})$ . It is now easy to see that  $k \leq 1$ .  $\blacksquare$

**Lemma 2.5.5** *If  $K \in H_{\infty,1}^c(\mathbb{C}^{\times m})$ ,  $K_1 \in H_\infty^c(\mathbb{C}^{p \times p})$  and  $K_2 \in H_\infty^c(\mathbb{C}^{m \times m_*})$ , then  $K_1KK_2 \in H_{\infty,1}^c(\mathbb{C}^{p \times m_*})$ .*

**Proof** From Lemma 2.5.2,  $K = NM^{-1}$ , where  $N, M \in \mathcal{MH}_\infty^c$  and  $\det(M) \in \mathcal{S}_\infty$  has, say  $k$  ( $\leq 1$ ) zeros in the open right half-plane. Thus

$$K_1KK_2 = \frac{1}{\det(M)} K_1 N \operatorname{adj}(M) K_2$$

and proceeding as in the proof of Lemma 2.5.4, we get that  $K_1KK_2 \in H_{\infty,k}^c(\mathbb{C}^{p \times m_*}) \subset H_{\infty,1}^c(\mathbb{C}^{p \times m_*})$ .  $\blacksquare$

**Lemma 2.5.6** *If  $K = NM^{-1} \in H_{\infty,[1]}^c(\mathbb{C}^{\times m})$ ,  $N \in H_\infty^c(\mathbb{C}^{\times n})$  and  $M \in H_\infty^c(\mathbb{C}^{m \times m})$ , with  $(M, N)$  right coprime, then  $\det(M)$  has exactly 1 zeros in  $\overline{\mathbb{C}_0^+}$ , and the zeros are all contained in the open right half-plane.*

**Proof** From Lemma 2.5.2, there exists another coprime factorization of  $K = N_*M_*^{-1}$ , where  $N_* \in H_\infty^c(\mathbb{C}^{\times m})$ ,  $M_* \in H_\infty^c(\mathbb{C}^{m \times m})$ , with  $(M_*, N_*)$  right coprime,

$M_*$  is rational and  $\det(M_*) \in \mathcal{R}_\infty$  has exactly 1 zeros in  $\overline{\mathbb{C}_0^+}$ , and the zeros are all contained in  $\mathbb{C}_0^+$ . From Lemma 2.5.3,  $M = M_*U$ , where  $U$  is an invertible element in  $\mathcal{MH}_\infty^c$ . From  $I = UU^{-1}$  we have that  $[\det(U)] [\det(U^{-1})] = 1$ , and hence  $\det(U(s)) \neq 0$  for  $s \in \overline{\mathbb{C}_0^+}$ . Since  $\det(M) = \det(M_*) \det(U)$ , it follows that  $\det(M)$  has exactly 1 zeros in  $\overline{\mathbb{C}_0^+}$ , and the zeros are all contained in the open right half-plane.  $\blacksquare$

The following technical lemma will be used in the characterization of solutions.

**Lemma 2.5.7** *If  $K(\cdot) \in \mathcal{MH}_\infty^c$ , then given any  $\epsilon > 0$ , there exists a  $\delta > 0$  such that whenever  $0 \leq \zeta \leq \delta$ , we have*

$$\|K(\zeta + i\cdot)\|_\infty \leq \|K(i\cdot)\|_\infty + \epsilon,$$

where  $\|\cdot\|_\infty$  denotes the  $L_\infty$ -norm (for a fixed  $\zeta \in \mathbb{R}$ ,  $\|K(\zeta + i\cdot)\|_\infty = \sup_{\omega \in \mathbb{R}} \|K(\zeta + i\omega)\|$ ).

In order to prove the above Lemma 2.5.7, we will use the following generalization of the so-called “3–lines theorem” to the matrix case. The statement and proof of the 3–lines theorem in the scalar case can be found, for instance, in Theorem 12.8, pages 257-258 of Rudin [70].

**Lemma 2.5.8** *Suppose*

1.  $\Omega = \{\zeta + i\omega \mid a < \zeta < b, \omega \in \mathbb{R}\}$ ,  $\overline{\Omega} = \{\zeta + i\omega \mid a \leq \zeta \leq b, \omega \in \mathbb{R}\}$ ,
2.  $K : \overline{\Omega} \rightarrow \mathbb{C}^{p \times m}$  is continuous on  $\overline{\Omega}$ ,
3.  $K$  is analytic in  $\Omega$ ,
4.  $\|K(s)\| \leq B$  for all  $s \in \Omega$  and for some fixed  $B < \infty$ .

If  $M(\zeta) = \sup\{\|K(\zeta + i\omega)\| \mid \omega \in \mathbb{R}\}$  for  $a \leq \zeta \leq b$ , then we have

$$M(\zeta)^{b-a} \leq M(a)^{b-\zeta} M(b)^{\zeta-a} \text{ for all } a < \zeta < b.$$

**Proof** 1. We assume first that  $M(a) = M(b) = 1$ . In this case, we have to prove that  $\|K(s)\| \leq 1$  for all  $s \in \Omega$ . For each  $\epsilon > 0$ , we define an auxiliary function

$$h_\epsilon(s) = \frac{1}{1 + \epsilon(s - a)} \text{ for all } s \in \overline{\Omega}. \quad (2.33)$$

Since  $\operatorname{Re}(1 + \epsilon(\zeta + i\omega - a)) = 1 + \epsilon(\zeta - a) \geq 1$  in  $\overline{\Omega}$ , we have

$$\|h_\epsilon(s)K(s)\| = |h_\epsilon(s)| \|K(s)\| \leq 1 \text{ for all } s \in \partial\Omega, \quad (2.34)$$

the boundary of  $\Omega$ . Also,  $|1 + \epsilon(\zeta + i\omega - a)| \geq \epsilon|\omega|$ , so that

$$\|h_\epsilon(s)K(s)\| = |h_\epsilon(s)| \|K(s)\| \leq \frac{B}{\epsilon|\omega|} \text{ for all } s = \zeta + i\omega \in \overline{\Omega} \setminus \mathbb{R}. \quad (2.35)$$

Let  $R$  be the rectangle cut off from  $\overline{\Omega}$  by the lines  $y = \pm \frac{B}{\epsilon}$  (see Figure 2.10).

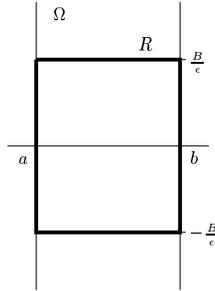


Figure 2.10: The rectangle  $R$ .

By (2.34) and (2.35),  $\|h_\epsilon K(s)\| \leq 1$  on  $\partial R$ , the boundary of the rectangle  $R$ . Hence  $\|h_\epsilon K(s)\| \leq 1$  on  $R$ , by the maximum modulus theorem (see for example, Theorem 3.18.4, page 115, Hille and Phillips [46]). But (2.35) shows that  $\|h_\epsilon K(s)\| \leq 1$  on the rest of  $\overline{\Omega}$ . Thus  $\|h_\epsilon K(s)\| \leq 1$  for all  $s \in \Omega$  and all  $\epsilon > 0$ . If we fix  $s \in \Omega$  and then let  $\epsilon$  tend to zero, we obtain the desired result.

2. We now turn to the general case with  $M(a) \neq 0$ . Let

$$f(s) = M(a)^{\frac{b-s}{b-a}} M(b)^{\frac{s-a}{b-a}},$$

where for  $M > 0$  and  $s$  complex,  $M^s$  is defined by

$$M^s = e^{s \log M},$$

and  $\log M$  is real. Then

1.  $f$  is analytic in  $\mathbb{C}$ ,
2.  $f$  has no zeros,
3.  $\frac{1}{f}$  is bounded in  $\overline{\Omega}$ ,
4.  $|f(a + i\omega)| = M(a)$ ,
5.  $|f(b + i\omega)| = M(b)$ ,

and so  $\frac{1}{f} K$  satisfies our previous assumptions. Thus  $\left\| \frac{1}{f(s)} K(s) \right\| \leq 1$  in  $\Omega$ , and hence

$$\|K(\zeta + i\omega)\| \leq |f(\zeta + i\omega)| = M(a)^{\frac{b-\zeta}{b-a}} M(b)^{\frac{\zeta-a}{b-a}}.$$

Hence  $M(\zeta)^{b-a} \leq M(a)^{b-\zeta} M(b)^{\zeta-a}$ .

3. Finally, we show that if  $M(a) = 0$ , then  $K$  is identically zero in  $\overline{\Omega}$ . Let

$$E_{11} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}_{p \times m}.$$

We have, for positive  $\epsilon$ ,

$$\sup_{\omega \in \mathbb{R}} \{ \|\epsilon E_{11} + K(a + i\omega)\| \} = \sup_{\omega \in \mathbb{R}} \{ \|\epsilon E_{11} + 0\| \} = \epsilon \|E_{11}\| (> 0),$$

and

$$\sup_{\omega \in \mathbb{R}} \{ \|\epsilon E_{11} + K(b + i\omega)\| \} \leq \sup_{\omega \in \mathbb{R}} \{ \epsilon \|E_{11}\| + \|K(b + i\omega)\| \} = \epsilon \|E_{11}\| + M(b).$$

Thus from the above, we obtain

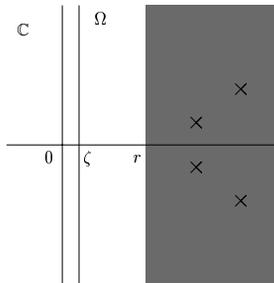
$$\|\epsilon E_{11} + K(\zeta + i\omega)\| \leq \epsilon^{\frac{b-\zeta}{b-a}} \|E_{11}\|^{\frac{b-\zeta}{b-a}} (\epsilon \|E_{11}\| + M(b))^{\frac{\zeta-a}{b-a}}.$$

This is true for all  $\epsilon > 0$ . Passing the limit as  $\epsilon$  tends to zero, we have

$$\|K(\zeta + i\omega)\| \leq 0 \|E_{11}\|^{\frac{b-\zeta}{b-a}} M(b)^{\frac{\zeta-a}{b-a}} = 0.$$

Consequently  $K(\zeta + i\omega) = 0$  for all  $\omega \in \mathbb{R}$  and  $a \leq \zeta < b$ . Finally the continuity of  $K$  yields that also  $K(b + i\omega) = 0$  for all  $\omega \in \mathbb{R}$ . Thus  $K$  is zero in  $\overline{\Omega}$ . ■

**Proof** (of Lemma 2.5.7.) Let  $K(s) = G(s) + F(s)$  where  $G$  is the matrix rational transfer operator of a system of MacMillan degree, say 1, with all its poles in the open right half-plane and  $F \in \mathcal{MH}_\infty$ . Let the poles of  $G$  be contained in the half-plane  $\mathbb{C}_r^+$  for some  $r > 0$ . Consider the function  $s \mapsto K(s)$  defined for  $s$  belonging to the infinite strip  $\overline{\Omega} := \{s \in \mathbb{C} \mid 0 \leq \text{Re}(s) \leq r\}$ . Clearly,  $K(\cdot)$  is continuous in  $\overline{\Omega}$  and analytic in the interior of  $\overline{\Omega}$ .



Using the triangle inequality, it is easy to see that  $K(\cdot)$  is bounded in  $\Omega$ :

1.  $s \mapsto G(s)$  is bounded in  $\Omega$  (since all its poles are in  $\mathbb{C}_r^+$ ) and
2.  $s \mapsto F(s)$  is bounded in  $\Omega$  (in fact, in  $\overline{\mathbb{C}_0^+}$ ).

For any  $\zeta > 0$ , define  $M(\zeta) = \sup_{\omega \in \mathbb{R}} \{\|K(\zeta + i\omega)\|\}$ . Using Lemma 2.5.8, we obtain

$$M(\zeta) \leq [\|K(i\cdot)\|_\infty]^{1-\frac{\zeta}{r}} M(r)^{\frac{\zeta}{r}}.$$

If  $\|K(i\cdot)\|_\infty = 0$ , then we have  $M(\zeta) = 0$ , and the result follows trivially. We now consider the case when  $\|K(i\cdot)\|_\infty \neq 0$ : we have

$$M(\zeta) \leq \|K(i\cdot)\|_\infty \left[ \frac{M(r)}{\|K(i\cdot)\|_\infty} \right]^{\frac{\zeta}{r}}.$$

Since  $\lim_{\zeta \rightarrow 0} \left[ \frac{M(r)}{\|K(i\cdot)\|_\infty} \right]^{\frac{\zeta}{r}} = 1$ , there exists a  $\delta$  such that  $0 < \delta < r$ , and for any  $\zeta$  satisfying  $0 \leq \zeta \leq \delta$ , we have  $\|K(\zeta + i\cdot)\|_\infty \leq \|K(i\cdot)\|_\infty + \epsilon$ . ■

**Lemma 2.5.9** *If  $F \in H_\infty^c(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$  and  $\lim_{\omega \rightarrow \pm\infty} F(i\omega)$  exists (say,  $F_\infty$ ), then*

$$\lim_{\substack{s \rightarrow \infty \\ s \in \mathbb{C}_0^+}} F(s)$$

*exists and equals  $F_\infty$ .*

**Proof** It follows from Theorem 5.18 (page 96, Rosenblum and Rovnyak [69]) that

$$F(\zeta + i\omega) = \frac{\zeta}{\pi} \int_{-\infty}^{\infty} \frac{1}{\zeta^2 + (t - \omega)^2} F(it) dt, \quad \zeta > 0.$$

Let  $\epsilon > 0$ . First we choose an  $R > 0$  (large enough) such that for all  $\omega$  in  $\mathbb{R}$  satisfying  $|\omega| > R$ , we have

$$\|F(i\omega) - F_\infty\| < \frac{\epsilon}{2}$$

We have

$$\begin{aligned} & \|F(\zeta + i\omega) - F_\infty\| \\ & \leq \left\| \frac{\zeta}{\pi} \int_{-\infty}^{\infty} \frac{1}{\zeta^2 + (t - \omega)^2} (F(it) - F_\infty) dt \right\| \\ & \leq \frac{\zeta}{\pi} \left\| \int_{-R}^R \frac{1}{\zeta^2 + (t - \omega)^2} (F(it) - F_\infty) dt \right\| + \frac{\epsilon}{2} \frac{\zeta}{\pi} \int_{-R}^R \frac{1}{\zeta^2 + (t - \omega)^2} dt \\ & \leq 2 \|F(i\cdot)\|_\infty \frac{\zeta}{\pi} \int_{-R}^R \frac{1}{\zeta^2 + (t - \omega)^2} dt + \frac{\epsilon}{2} \cdot 1, \end{aligned}$$

where we have used

$$\frac{\zeta}{\pi} \int_{-\infty}^{\infty} \frac{1}{\zeta^2 + (t - \omega)^2} dt = 1.$$

We note that

$$\frac{\zeta}{\pi} \int_{-R}^R \frac{1}{\zeta^2 + (t - \omega)^2} dt = \frac{1}{\pi} \operatorname{Arctan} \left[ \frac{2R\zeta}{\zeta^2 + \omega^2 - R^2} \right]$$

and so it is clear that there exists a  $R_*$  ( $> R$ ) such that all  $\zeta + i\omega$  satisfying  $|\zeta + i\omega| > R$ , we have

$$\frac{\zeta}{\pi} \int_{-R}^R \frac{1}{\zeta^2 + (t - \omega)^2} dt < \frac{\epsilon}{4 \|F(i \cdot)\|_{\infty}}.$$

Consequently, for all  $s \in \overline{\mathbb{C}_0^+}$  with  $|s| > R_*$ , we have

$$\|F(\zeta + i\omega) - F_{\infty}\| \leq 2 \|F(i \cdot)\|_{\infty} \frac{\epsilon}{4 \|F(i \cdot)\|_{\infty}} + \frac{\epsilon}{2} = \epsilon$$

and this completes the proof. ■

**Lemma 2.5.10** *If*

1.  $F(\cdot) \in H_{\infty}^c(\mathbb{C}^{k \times k})$
2. for every  $\omega \in \mathbb{R}$ ,  $F(i\omega)$  is invertible
3.  $\lim_{\omega \rightarrow \pm\infty} F(i\omega)$  exists (say,  $F_{\infty}$ ) and  $F_{\infty}$  is invertible

then  $F(\cdot)^{-1} \in H_{\infty, \bullet}^c(\mathbb{C}^{k \times k})$ .

**Proof** From Lemma 2.5.9 above, we have

$$\lim_{\substack{s \rightarrow \infty \\ s \in \mathbb{C}_0^+}} F(s) = F_{\infty}.$$

We prove that  $s \mapsto \det(F(s))$  has a finite number of zeros, which are all contained in the open right half-plane. Since  $\det(\cdot) : \mathbb{C}^{k \times k} \rightarrow \mathbb{C}$  is a continuous function, it follows that

$$\lim_{\substack{s \rightarrow \infty \\ s \in \mathbb{C}_0^+}} \det(F(s)) = \det(F_{\infty}) \neq 0.$$

Consequently, there exists a  $R > 0$  large enough such that  $\det(F(s)) \neq 0$  for every  $s \in \overline{\mathbb{C}_0^+}$  with  $|s| > R$ . The function  $s \mapsto \det(F(s))$  is analytic in  $\Omega =$

$\{s \in \mathbb{C}_0^+ \mid |s| < R\}$ . But it is not identically 0 in  $\Omega$  (for otherwise, owing to the continuity of  $s \mapsto \det(F(s))$  in  $\overline{\Omega} = \{s \in \overline{\mathbb{C}_0^+} \mid |s| \leq R\}$  we would have  $\det(F(i\omega)) = 0$  for all  $\omega \in [-iR, iR]$ , a contradiction!). Thus the zeros of  $s \mapsto \det(F(s))$  have no accumulation point in  $\Omega$ . But there are no zeros on  $[-iR, iR] \cup \{s \in \overline{\mathbb{C}_0^+} \mid |s| = R\}$ , and so the zeros of  $s \mapsto \det(F(s))$  have no accumulation point in  $\overline{\Omega}$ . Since the set  $\overline{\Omega}$  is compact, it now follows that there are only finitely many zeros. Furthermore, they are all contained in  $\mathbb{C}_0^+$ . Finally, the result follows using Lemma 2.5.4. ■

## Chapter 3

# Compactness and nuclearity of Hankel operators

In this chapter, we examine the relationships between the exponential (or strong) stability of certain classes of well-posed linear systems and the compactness and nuclearity properties of the Hankel operator. New sufficient conditions for nuclearity are given for exponentially stable, regular, well-posed linear systems with an analytic semigroup. In the last section of this chapter we will derive a bound on the  $L_\infty$ -error of a sub-optimal Hankel norm approximant of a system with a nuclear Hankel operator.

We consider the Hankel operator of stable matrix-valued functions  $G \in H_\infty(\mathbb{C}^{p \times m})$ . The Hankel operator with symbol  $G$  is the operator  $H_G : H_2(\mathbb{C}^m) \rightarrow H_2(\mathbb{C}^p)$  given by

$$H_G f = \Pi(\Lambda_G f_-) \quad \text{for } f \in H_2(\mathbb{C}^m), \quad (3.1)$$

where  $\Lambda_G$  is the multiplication map on  $L_2(i\mathbb{R}, \mathbb{C}^m)$  induced by  $G$  (see Theorem A.6.26, page 647, Curtain and Zwart [27]),  $\Pi$  is the orthogonal projection operator from  $L_2(i\mathbb{R}, \mathbb{C}^p)$  onto  $H_2(\mathbb{C}^p)$  and  $f_-(s) := f(-s)$ .

Practical control design is typically based on a reduced-order model of the original system. Many design methodologies utilize a rational approximation of a stable transfer function in the  $L_\infty$ -norm (for example, see Chapter 9, pages 457-563, Curtain and Zwart [27]). For this to be possible the Hankel operator with symbol  $G \in H_\infty(\mathbb{C}^{p \times m})$  should be compact. We quote the following criterion (see for instance Corollary 4.10, page 46, Partington [62]):

**Theorem 3.0.11** (Hartman)  *$G \in L_\infty(i\mathbb{R}, \mathbb{C}^{p \times m})$  determines a compact Hankel operator iff  $G_-(\cdot) \in H_\infty(\mathbb{C}^{p \times m}) + \mathcal{C}_0(i\mathbb{R}, \mathbb{C}^{p \times m})$ , where  $\mathcal{C}_0(i\mathbb{R}, \mathbb{C}^{p \times m})$  denotes the*

space of continuous  $p \times m$  complex matrix-valued functions defined on  $i\mathbb{R}$ , with a (unique) limit at  $\pm i\infty$ .

Since most models of infinite-dimensional systems are obtained, not as transfer functions, but as realizations, we are interested in deducing the properties of the Hankel operator from properties of the realization. To do this we introduce the time-domain Hankel operator, which is defined in terms of  $h$ , the inverse-Laplace transform of  $G$ . If  $h \in L_1([0, \infty), \mathbb{C}^{p \times m})$  or  $L_2([0, \infty), \mathbb{C}^{p \times m})$ , we define the time-domain Hankel operator  $\Gamma_h : L_2([0, \infty), \mathbb{C}^m) \rightarrow L_2([0, \infty), \mathbb{C}^p)$  by

$$(\Gamma_h u)(t) = \int_0^\infty h(t+s)u(s)ds, \quad t \geq 0, \quad (3.2)$$

for all  $u \in L_2([0, \infty), \mathbb{C}^m)$ . In the case that  $h \in L_1([0, \infty), \mathbb{C}^{p \times m})$ , it is well-known that  $\Gamma_h$  is compact (see for example Lemma 8.2.4, page 399, Curtain and Zwart [27]) and so  $\Gamma_h$  has countably many singular values (square roots of the eigenvalues of  $\Gamma_h^* \Gamma_h$ )  $\sigma_1 \geq \sigma_2 \geq \dots \geq 0$  and these are also called the *Hankel singular values* of  $G$ . If  $h \in L_2([0, \infty), \mathbb{C}^{p \times m})$ , then (3.2) may not be in  $L_2([0, \infty), \mathbb{C}^p)$ . If, however, we also assume that  $G \in L_\infty(i\mathbb{R}, \mathbb{C}^{p \times m})$ , then (3.2) always defines  $\Gamma_h$  as a bounded operator from  $L_2([0, \infty), \mathbb{C}^m)$  to  $L_2([0, \infty), \mathbb{C}^p)$  (see Proposition 8, page 224, Keulen [83]). In either of these cases,  $\Gamma_h$  is isomorphic to  $H_G$  under the Laplace (or Fourier) transform (see Lemma 8.2.3, page 397, Curtain and Zwart [27] and Keulen [83]).

Let us consider two classes of systems; the first was the subject of the book [27] by Curtain and Zwart and the second was the main topic of the recent book by Oostveen [61].

**Class 1.**  $\Sigma(A, B, C)$  where  $A$  is the infinitesimal generator of an exponentially stable strongly continuous semigroup  $\{T(t)\}_{t \geq 0}$  on a separable Hilbert space  $X$ ,  $B \in \mathcal{L}(\mathbb{C}^m, X)$  and  $C \in \mathcal{L}(X, \mathbb{C}^p)$ .

For this class, it is well-known and easy to verify that the impulse response is  $h(t) = CT(t)B$ ,  $h \in L_1([0, \infty), \mathbb{C}^{p \times m})$ , and the Hankel operator is compact.

**Class 2.**  $\Sigma(A - BB^*, B, B^*)$ , where  $A$  generates a contraction semigroup on the Hilbert space  $X$ ,  $B \in \mathcal{L}(\mathbb{C}^m, X)$ .

For this class, it is known that  $G(s) = B^*(sI - A + BB^*)^{-1}B \in H_\infty(\mathbb{C}^{m \times m})$ ,  $h \in L_2([0, \infty), \mathbb{C}^{m \times m})$ , but  $h \notin L_1([0, \infty), \mathbb{C}^{m \times m})$  in general (see Curtain and Zwart [28]). So by Proposition 8 (page 224, Keulen [83]) it follows that the time-domain Hankel operator is well-defined by (3.2).  $A - BB^*$  in general does not generate an exponentially stable semigroup, but under mild conditions it does generate a *strongly stable semigroup*  $\{T_B(t)\}_{t \geq 0}$  (that is,  $T_B(t)x \rightarrow 0$  as  $t \rightarrow \infty$  for all  $x \in X$ ).

Since the Hankel operator of Class 1 is compact, one wonders if relaxing exponential stability to strong stability is still sufficient for compactness. Surprisingly, the reverse statement holds if  $A$  and  $A^*$  are skew-symmetric (that is,  $\langle x, Ax \rangle + \langle Ax, x \rangle = 0$  for all  $x \in D(A)$  and  $\langle x, A^*x \rangle + \langle A^*x, x \rangle = 0$  for all  $x \in D(A^*)$ ). In section 3.1 we prove that if  $A$  and  $A^*$  are skew-symmetric,  $\{T_B(t)\}_{t \geq 0}$  and  $\{T_B(t)^*\}_{t \geq 0}$  are strongly stable but not exponentially stable, and  $X$  is infinite-dimensional, then  $\Gamma_h$  will never be compact! On the other hand, by means of an example, we show that systems from Class 2 can have a compact Hankel operator even if  $\{T_B(t)\}_{t \geq 0}$  is not strongly stable. In other words, the stability of the semigroup says little about the compactness of the Hankel operator.

While compactness of the Hankel operator is essential to obtain a rational approximation in the  $H_\infty$ -norm, the property of nuclearity is desirable, because it gives a-priori error bounds in terms of the singular values. We recall that a compact Hankel operator is *nuclear* if  $\sum_{k=1}^\infty \sigma_k < \infty$ , where  $\sigma_k$ 's are the singular values in decreasing order. (For background material about nuclearity, we refer to Chapter 1 of Partington [62].) In Glover et al. [37] it was shown that if  $h \in L_1([0, \infty), \mathbb{C}^{\times m}) \cap L_2([0, \infty), \mathbb{C}^{\times m})$ , and  $\Gamma_h$  is nuclear, then for a given integer 1, there exists a truncated balanced realization  $G_{b,1}$  and an optimal Hankel-norm approximation  $G_{0,1}$ , both of order 1, such that

$$\begin{aligned} \|G - G_{b,1}\|_\infty &\leq 2 \cdot \sum_{k=1+1}^\infty \sigma_k, \text{ and} \\ \|G - G_{0,1}\|_\infty &\leq \sum_{k=1+1}^\infty \sigma_k. \end{aligned}$$

In the last section of this chapter we will also find a bound on the  $L_\infty$ -error of any sub-optimal Hankel norm approximant of a transfer function for which the Hankel operator is nuclear.

There exist various known conditions for nuclearity in terms of the transfer functions.

**Theorem 3.0.12** (Coifman and Rochberg [11].) *If  $G \in H_\infty(\mathbb{C}^{\times m})$ , then  $H_G$  is nuclear iff*

$$\int_{-\infty}^\infty \int_0^\infty \|G''(x + iy)\| \, dx dy < \infty. \quad (3.3)$$

**Theorem 3.0.13** (Coifman and Rochberg [11]) *The Hankel operator given by (3.2) is nuclear iff  $G$  possesses on  $\mathbb{C}_0^+$  an expansion of the form*

$$G(s) = \sum_{n=1}^\infty \frac{1}{s - a_n} G_n \quad (3.4)$$

where  $G_n \in \mathbb{C}^{p \times m}$  and  $a_n \in \mathbb{C}_0^-$  are such that

$$\sum_{n=1}^{\infty} \frac{\|G_n\|}{|Re(a_n)|} < \infty. \quad (3.5)$$

These theorems made it possible to obtain sufficient conditions for the nuclearity of transfer functions of delay systems (see Zwart et al. [94] and Glover et al. [38]). See also the recent results on fractional transfer functions in Bonnet and Partington [9]. In this chapter we give sufficient conditions for nuclearity in terms of the state-space realizations.

Our new results on nuclearity in Section 3.2 are

1. Class 1 has a nuclear Hankel operator (Theorem 3.2.1).
2. A class of regular well-posed linear systems with an exponentially stable analytic semigroup and unbounded  $B$  and  $C$  has a nuclear Hankel operator (Theorem 3.2.3).

Analytic semigroups are generated by parabolic partial differential operators and hyperbolic partial differential operators with structural damping (see Pazy [63]). Consequently, Theorem 3.2.3 has important consequences for model reduction of distributed systems with an analytic semigroup with unbounded sensing and control.

## 3.1 Compactness of Hankel operators

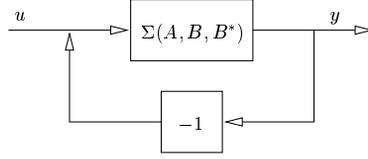
In the introduction to this chapter, we recalled that linear systems with bounded, finite-rank input and output operators, and an exponentially stable semigroup always have a compact Hankel operator. One might hope that this would also be the case if we only have strong stability. However, by means of two examples, we show that the property of strong stability does not imply the compactness of the Hankel operator.

Let us consider Class 2. This class arises by stabilizing the open-loop system  $\Sigma(A, B, B^*)$  with transfer function  $G_0(s) = B^*(sI - A)^{-1}B$ , via the static output feedback  $u = -y$ ,

which results in the closed-loop system  $\Sigma(A - BB^*, B, B^*)$  with transfer function

$$G(s) = G_0(s) (I + G_0(s))^{-1} = B^*(sI - A + BB^*)^{-1}B.$$

This closed-loop system has several nice properties (see Curtain and Zwart [28]):



**C1.**  $A - BB^*$  generates a contraction semigroup  $\{T_B(t)\}_{t \geq 0}$ .

**C2.**  $\int_0^\infty \|B^*T_B(t)x\|^2 dt \leq \frac{1}{2}\|x\|^2$ .

**C3.**  $\int_0^\infty \|B^*T_B(t)^*x\|^2 dt \leq \frac{1}{2}\|x\|^2$ .

**C4.**  $G(s) = B^*(sI - A + BB^*)^{-1}B \in H_\infty(\mathcal{L}(U))$ .

C2 and C3 show that the system has an impulse response  $h(\cdot) = B^*T_B(\cdot)B \in L_2([0, \infty), \mathcal{L}(U))$  and bounded observability and controllability maps  $\mathcal{C}$  and  $\mathcal{B}$  defined as follows:

1.  $\mathcal{B} : L_2([0, \infty), U) \rightarrow X$  is defined by

$$\mathcal{B}u = \int_0^\infty T_B(t)Bu(t)dt \text{ for all } u \in L_2([0, \infty), U). \quad (3.6)$$

2.  $\mathcal{C} : X \rightarrow L_2([0, \infty), Y)$  is defined by

$$(\mathcal{C}x)(t) = B^*T_B(t)x \text{ for all } t \geq 0 \text{ and all } x \in X. \quad (3.7)$$

The semigroup  $\{T_B(t)\}_{t \geq 0}$  is not necessarily strongly stable. Sufficient conditions for  $\{T_B(t)\}_{t \geq 0}$  and  $\{T_B(t)^*\}_{t \geq 0}$  to be strongly stable can be found in Arendt and Batty [3]:

**N1.** The intersection of the spectrum of  $A$  with the imaginary axis is at most countable.

OR

**N2.**  $\{x \in X \mid B^*T(t)x = 0, \|T(t)^*x\| = \|x\| = \|T(t)x\| \ \forall t \geq 0\} = \{0\}$ .

Note that in N2,  $B^*T(t)x = 0$  can be replaced by  $B^*T(t)^*x = 0$ , and if  $\Sigma(A, B, B^*)$  is approximately controllable or observable, then N2 holds.

The Hankel operator  $\Gamma_h$  is equal to  $\mathcal{C}\mathcal{B}$ . The controllability Gramian  $L_B = \mathcal{B}\mathcal{B}^*$  and the the observability Gramian  $L_C = \mathcal{C}^*\mathcal{C}$  always satisfy their respective Lyapunov equations:



Let  $\{x_m\}_{m \geq 1}$  be a sequence in  $D(A)$  and let  $x_m \rightarrow x_0$  and  $Ax_m \rightarrow y_0$  as  $m \rightarrow \infty$ . Since the sequence  $\{Ax_m\}_{m \geq 1}$  is bounded, there exists a  $M > 0$  such that

$$\sum_{k=1}^{\infty} \left( |-\mathbf{k}\langle x_m, e_{2k-1} \rangle|^2 + |\mathbf{k}\langle x_m, e_{2k} \rangle|^2 \right) < M, \text{ for all } m \geq 1.$$

Consequently, for any  $N \in \mathbb{N}$ ,

$$\sum_{k=1}^N \left( |-\mathbf{k}\langle x_m, e_{2k-1} \rangle|^2 + |\mathbf{k}\langle x_m, e_{2k} \rangle|^2 \right) < M, \text{ for all } m \geq 1.$$

Owing to the continuity of the inner product and the fact that  $x_m \rightarrow x_0$ , we obtain

$$\sum_{k=1}^N \left( |-\mathbf{k}\langle x_0, e_{2k-1} \rangle|^2 + |\mathbf{k}\langle x_0, e_{2k} \rangle|^2 \right) \leq M.$$

Since the choice of  $N$  was arbitrary, it follows that

$$\sum_{k=1}^{\infty} \left( |-\mathbf{k}\langle x_0, e_{2k-1} \rangle|^2 + |\mathbf{k}\langle x_0, e_{2k} \rangle|^2 \right) \leq M.$$

Consequently,  $x_0 \in D(A)$  with  $Ax_0 = y_0$  and so  $A$  is closed.

In fact it can be easily checked that  $A$  is a Riesz spectral operator with the (totally disconnected) set of simple unstable eigenvalues  $\{\pm ni\}_{n \in \mathbb{N}}$  (see Figure 3.1) and the corresponding (orthogonal) Riesz basis of eigenvectors  $\left\{ \frac{1}{\sqrt{2}}(e_n \pm ie_{n+1}) \right\}_{n \in \mathbb{N}}$ .

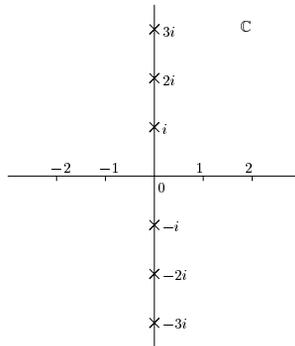


Figure 3.1:  $\sigma(A)$ .

Since  $Ax + A^*x = 0$  for all  $x \in D(A) = D(A^*)$ , it follows that  $A$  is the infinitesimal generator of a contraction strongly continuous semigroup on the

Hilbert space  $\ell_2(\mathbb{N})$ . We have  $0 \in \rho(A)$  and

$$A^{-1} = \left[ \begin{array}{ccc} \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} & & \\ & \boxed{\begin{matrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{matrix}} & \\ & & \boxed{\begin{matrix} 0 & -\frac{1}{3} \\ \frac{1}{3} & 0 \end{matrix}} \\ & & & \ddots \end{array} \right] \in \mathcal{L}(\ell_2(\mathbb{N})).$$

It can be easily seen that  $R_n \rightarrow A^{-1}$  as  $n \rightarrow \infty$  in the uniform operator topology, where

$$R_n := \left[ \begin{array}{ccc} \boxed{\begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix}} & & \\ & \ddots & \\ & & \boxed{\begin{matrix} 0 & -\frac{1}{n} \\ \frac{1}{n} & 0 \end{matrix}} \\ & & & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \\ & & & & \ddots \end{array} \right] \in \mathcal{L}(\ell_2(\mathbb{N})).$$

Since each  $R_n$  has finite rank  $2n$ ,  $A^{-1}$  is the uniform limit of a sequence of compact operators and so  $A^{-1}$  is compact (see for example Theorem A.3.22.e, page 587, Curtain and Zwart [27]). From Theorem 6.29 (page 187, Kato [49]) it follows that  $A$  has compact resolvent.

Let  $B \in \mathcal{L}(\mathbb{C}, \ell_2(\mathbb{N}))$  be defined by

$$B = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \\ 0 \\ \frac{1}{3} \\ 0 \\ \vdots \end{bmatrix}.$$

It follows from Theorem 4.2.3 (page 164, Curtain and Zwart [27]) that  $\Sigma(A, B, -)$  is approximately controllable. Dually,  $\Sigma(A^*, -, B^*)$  is approximately observable. Consequently, from Lemma 2.2.6 (page 23, Oostveen [60]) it follows that  $A_B := A - BB^*$  and  $A_B^* = A^* - BB^*$  generate strongly stable semigroups  $\{T_B(t)\}_{t \geq 0}$  and  $\{T_B(t)^*\}_{t \geq 0}$ , respectively, on  $\ell_2(\mathbb{N})$ .



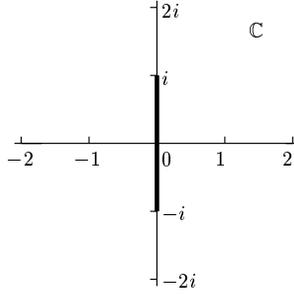


Figure 3.2:  $\sigma(A) = \sigma_c(A) = [-i, i]$ .

Claim:  $\Sigma(A, B, B^*)$  realizes the transfer function  $\frac{1}{\sqrt{s^2+1}}$ .

We know that  $s \mapsto B^*(sI - A)^{-1}B$  is analytic in  $\mathbb{C} \setminus [-i, i]$ . Moreover for  $\omega > 1$ , it can be checked easily that  $B^*(i\omega - A)^{-1}B = \frac{-i}{\sqrt{\omega^2-1}}$ , as follows. Let  $L_2(\mathbb{T})$  be the the space of square integrable functions on the unit circle and let  $L_\varphi$  be the multiplication operator corresponding to the bounded function

$$\varphi(z) = -\frac{1}{2}z^{-1} + i\omega + \frac{1}{2}z.$$

From Exercise 241 (page 135, Halmos [44]), it follows that the bilaterally infinite matrix

$$i\omega I - A = \begin{bmatrix} \ddots & \ddots & & & & \\ & \ddots & & & & \\ & & i\omega & \frac{1}{2} & & \\ & & -\frac{1}{2} & i\omega & \frac{1}{2} & \\ & & & -\frac{1}{2} & i\omega & \ddots \\ & & & & \ddots & \ddots \end{bmatrix}$$

is the Laurent matrix corresponding to the multiplication operator  $L_\varphi$  on  $L_2(\mathbb{T})$  with respect to the familiar standard orthonormal basis in  $L_2(\mathbb{T})$ , namely  $\{e_n(z) = z^n\}_{n \in \mathbb{Z}}$ . Thus  $(i\omega I - A)^{-1}$  corresponds to multiplication by

$$\begin{aligned} \frac{1}{\varphi(z)} &= \frac{1}{-\frac{1}{2}z^{-1} + i\omega + \frac{1}{2}z} \\ &= \frac{1}{i\omega \left[1 - \frac{1}{2i\omega} \left(-z + \frac{1}{z}\right)\right]} \\ &= \frac{1}{i\omega} \left[1 + \frac{1}{2i\omega} \left(-z + \frac{1}{z}\right) + \left[\frac{1}{2i\omega} \left(-z + \frac{1}{z}\right)\right]^2 + \dots\right]. \end{aligned} \tag{3.8}$$

Since we want to find out  $B^*(i\omega I - A)^{-1}B$ , we are actually finding the  $(0, 0)^{th}$

entry of  $(i\omega I - A)^{-1}$ , which is the coefficient of  $z^0$  in (3.8): namely,

$$\frac{1}{i\omega} \sum_{n=0}^{\infty} \left(\frac{1}{2i\omega}\right)^{2n} (-1)^n \binom{2n}{n}.$$

But for  $\omega > 1$ , we have

$$\begin{aligned} \frac{-i}{\sqrt{\omega^2 - 1}} &= \frac{1}{i\omega} \left(1 - \frac{1}{\omega^2}\right)^{-\frac{1}{2}} \\ &= \frac{1}{i\omega} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n \left(\frac{1}{\omega^2}\right)^n \\ &= \frac{1}{i\omega} \sum_{n=0}^{\infty} \left(\frac{1}{2i\omega}\right)^{2n} (-1)^n 2^{2n} \binom{-\frac{1}{2}}{n} (-1)^n. \end{aligned}$$

It is easy to see that

$$2^{2n} \binom{-\frac{1}{2}}{n} (-1)^n = \binom{2n}{n},$$

and so it follows that for  $\omega > 1$ ,  $B^*(i\omega - A)^{-1}B = \frac{-i}{\sqrt{\omega^2 - 1}}$ .

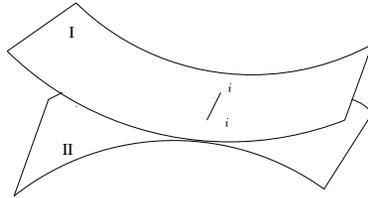


Figure 3.3: The Riemann surface of the map  $s \mapsto \frac{1}{\sqrt{s^2+1}}$ .

Consider the map  $s \mapsto \frac{1}{\sqrt{s^2+1}}$ . We shall take two copies of the  $s$ -plane, with cuts along the segment  $[-i, i]$ . The formation of an analytic branch of the map  $s \mapsto \frac{1}{\sqrt{s^2+1}}$  takes place in each of these planes. Now we shall join the edges of our cuts cross-wise with the help of two segments  $[-i, i]$ , the inner points of which will be considered to be different, although geometrically they coincide. This is the Riemann surface of the function  $s \mapsto \frac{1}{\sqrt{s^2+1}}$  (see Figure 3.3). It is double-sheeted, with two branch points on  $i$  and  $-i$ . The function  $s \mapsto \frac{1}{\sqrt{s^2+1}}$  is single-valued and continuous on this surface. We consider the analytic branch of  $s \mapsto \frac{1}{\sqrt{s^2+1}}$  corresponding to the plane I;  $i\omega \mapsto \frac{-i}{\sqrt{\omega^2-1}}$  for  $\omega > 1$ . Since  $G_0(s)$  and this analytic branch match on  $\{i\omega \mid \omega > 1\}$ , the claim follows.

Thus  $\Sigma(A - BB^*, B, B^*)$  realizes the transfer function  $G(s) = \frac{1}{1+\sqrt{s^2+1}}$ . Since  $G(s) = \frac{1}{1+\sqrt{s^2+1}}$ , it can be easily checked that  $G(s) \in H_\infty(\mathbb{C})$ . Moreover,  $\omega \mapsto G(i\omega)$  is continuous and has the limit 0 as  $\omega \rightarrow \pm\infty$ . So it follows from

Theorem 3.0.11, that  $G$  determines a compact Hankel operator. However, our earlier discussion shows that in this case  $A - BB^*$  cannot generate a strongly stable semigroup.  $\diamond$

## 3.2 Nuclearity of Hankel operators

In this section we study the nuclearity property of the Hankel operator of a system in terms of a given realization  $(A, B, C)$ .

First we show that an exponentially stable state linear system with bounded inputs and outputs and finite-dimensional input and output spaces is nuclear.

**Theorem 3.2.1** *Let  $A$  be the infinitesimal generator of an exponentially stable strongly continuous semigroup  $\{T(t)\}_{t \geq 0}$  on the separable Hilbert space  $X$ , with  $B \in \mathcal{L}(\mathbb{C}^m, X)$ ,  $C \in \mathcal{L}(X, \mathbb{C}^p)$ . Then*

1. *The observability operator  $\mathcal{C} : X \rightarrow L_2([0, \infty), \mathbb{C}^p)$  defined by  $(\mathcal{C}x)(\cdot) = CT(\cdot)x$  is Hilbert-Schmidt.*
2. *The controllability operator  $\mathcal{B} : L_2([0, \infty), \mathbb{C}^m) \rightarrow X$  defined by  $\mathcal{B}u = \int_0^\infty T(t)Bu(t)dt$  is Hilbert-Schmidt.*
3.  *$L_C = C^*C$ ,  $L_B = BB^*$  and  $\Gamma_h = \mathcal{C}\mathcal{B}$  are all nuclear.*

**Proof** 1. Define  $\mathcal{C}_k : X \rightarrow L_2(0, \infty)$ ,  $k \in \{1, \dots, p\}$  by

$$(\mathcal{C}_k x)(t) = \langle CT(t)x, e_k \rangle = \langle x, T(t)^* C^* e_k \rangle,$$

where  $\{e_1, \dots, e_p\}$  is the standard basis for  $\mathbb{C}^p$ . We have

$$\begin{aligned} |(\mathcal{C}_k x)(t)| &= |\langle x, T(t)^* C^* e_k \rangle| \\ &\leq \|x\| \|T(t)^* C^* e_k\| \leq \|x\| \|T(t)^*\| \|C^*\| \|e_k\| \\ &\leq \|x\| M e^{-\alpha t} \|C^*\|, \end{aligned}$$

and  $\int_0^\infty M e^{-\alpha t} \|C^*\|^2 dt < \infty$ . We will now use the following result which is an adaptation of Theorem 6.12 (page 140, Weidmann [85]):

**Theorem 3.2.2** *Let  $K$  be a bounded linear operator from a Hilbert space  $\mathcal{H}$  into  $L_2(0, \infty)$ . If there exists a function  $\kappa \in L_2(0, \infty)$  such that  $|(Kv)(t)| \leq \kappa(t)\|v\|$  for almost all  $t \in (0, \infty)$ , and all  $v \in \mathcal{H}$ , then  $K$  is Hilbert-Schmidt.*

Applying this result we obtain that  $\mathcal{C}_k$  is Hilbert-Schmidt. Consequently, for an arbitrary orthonormal basis  $\{x_k\}$  of  $X$ ,

$$\sum_{j=1}^{\infty} \|\mathcal{C}_k x_j\|_{L_2(0,\infty)}^2 < \infty \text{ for all } k \in \{1, \dots, p\},$$

and

$$\sum_{k=1}^p \sum_{j=1}^{\infty} \|\mathcal{C}_k x_j\|_{L_2(0,\infty)}^2 < \infty.$$

Thus

$$\sum_{j=1}^{\infty} \|\mathcal{C} x_j\|_{L_2([0,\infty), \mathbb{C}^p)}^2 < \infty,$$

and so  $\mathcal{C}$  is Hilbert-Schmidt.

2. From Theorem 6.9 (Weidmann [85]), it follows that  $\mathcal{B}$  is Hilbert-Schmidt iff  $\mathcal{B}^*$  is Hilbert-Schmidt. But  $\mathcal{B}^*$  is Hilbert-Schmidt by applying the first part of the lemma to the dual system  $\Sigma(A^*, C^*, B^*)$ .

3. Using Theorem 7.10(b) (Weidmann [85]), we obtain that  $L_C = C^*C$ ,  $L_B = \mathcal{B}\mathcal{B}^*$  and  $\Gamma_h = \mathcal{C}\mathcal{B}$  are all nuclear. ■

We remark that the Hilbert-Schmidt property of  $\mathcal{C}$  was already shown in Dumortier [29] (page 24, Proposition 1.0.2).

We now show that a class of regular linear systems with an exponentially stable analytic semigroup and unbounded  $B$  and  $C$  has a nuclear Hankel operator. We use the notation introduced in Section 2.4.

**Theorem 3.2.3** *Let*

1.  $-A$  generate an exponentially stable, analytic, strongly continuous semigroup  $\{T(t)\}_{t \geq 0}$  on  $Z$ ,
2.  $B \in \mathcal{L}(\mathbb{C}^m, Z_{\alpha_B})$ , and
3.  $C \in \mathcal{L}(Z_{\alpha_C}, \mathbb{C}^p)$ ,

where  $\alpha_B \leq \alpha_C < \alpha_B + 1$ . If  $\gamma$  satisfies  $\alpha_C - \frac{1}{2} < \gamma < \alpha_B + \frac{1}{2}$ , then

1.  $-A, B, C$  generate a regular linear system with state space  $Z_\gamma$ , input space  $\mathbb{C}^m$  and output space  $\mathbb{C}^p$ . The transfer function is given by  $G(s) = C(sI + A)^{-1}B$ , and it satisfies

$$\lim_{\substack{|s| \rightarrow \infty \\ s \in \mathbb{C}_0^+}} \|G(s)\| = 0.$$

2. The controllability map  $\mathcal{B} \in \mathcal{L}(L_2([0, \infty), \mathbb{C}^m), Z_\gamma)$  and the observability map  $\mathcal{C} \in \mathcal{L}(Z_\gamma, L_2([0, \infty), \mathbb{C}^p))$  are Hilbert-Schmidt operators.
3.  $h(\cdot) := CT(\cdot)B \in L_1([0, \infty), \mathbb{C}^{p \times m})$  and the Hankel operator  $\Gamma_h$  satisfies  $\Gamma_h = \mathcal{C}\mathcal{B}$ , where  $\mathcal{B} \in \mathcal{L}(L_2([0, \infty), \mathbb{C}^m), Z_\gamma)$  and  $\mathcal{C} \in \mathcal{L}(Z_\gamma, L_2([0, \infty), \mathbb{C}^p))$ . Furthermore,  $\Gamma_h$  is nuclear.

**Proof** 1. This part follows from Staffans [79] (page 251) or Staffans [77].

2. Just as in Theorem 3.2.1, it is sufficient to prove this for the case  $p = 1$ . From part 1 above, we know that  $\mathcal{C}$  is an admissible observation operator for  $\{T(t)\}_{t \geq 0}$  with state space  $Z_\gamma$  and so  $\mathcal{C} \in \mathcal{L}(Z_\gamma, L_2([0, \infty), \mathbb{C}))$ . Since  $C \in \mathcal{L}(Z_{\alpha_C}, \mathbb{C})$ , for every  $z \in Z_{\alpha_C}$ , we have

$$\begin{aligned} |(\mathcal{C}z)(t)| &= |CT(t)z| \\ &\leq \|C\|_{\mathcal{L}(Z_{\alpha_C}, \mathbb{C})} \|T(t)z\|_{Z_{\alpha_C}} \\ &= \|C\|_{\mathcal{L}(Z_{\alpha_C}, \mathbb{C})} \|A^{\alpha_C - \gamma} T(t) A^\gamma z\|_Z \\ &\leq \|C\|_{\mathcal{L}(Z_{\alpha_C}, \mathbb{C})} \|A^{\alpha_C - \gamma} T(t)\|_{\mathcal{L}(Z)} \|A^\gamma z\|_Z \\ &= \|C\|_{\mathcal{L}(Z_{\alpha_C}, \mathbb{C})} \|A^{\alpha_C - \gamma} T(t)\|_{\mathcal{L}(Z)} \|z\|_{Z_\gamma}. \end{aligned}$$

**Case 1:** If  $\alpha_C - \gamma \geq 0$ , then we have

$$\begin{aligned} |(\mathcal{C}z)(t)| &= \|C\|_{\mathcal{L}(Z_{\alpha_C}, \mathbb{C})} \|A^{\alpha_C - \gamma} T(t)\|_{\mathcal{L}(Z)} \|z\|_{Z_\gamma} \\ &\leq \|C\|_{\mathcal{L}(Z_{\alpha_C}, \mathbb{C})} K_1 \frac{e^{-\epsilon t}}{t^{\alpha_C - \gamma}} \|z\|_{Z_\gamma}, \end{aligned}$$

and  $t \mapsto \frac{e^{-\epsilon t}}{t^{\alpha_C - \gamma}} \in L_2([0, \infty), \mathbb{C})$ , for  $\gamma > \alpha_C - \frac{1}{2}$ . So by Theorem 6.12 (page 140, Weidmann [85]), it follows that  $\mathcal{C}$  is Hilbert-Schmidt for all  $\gamma$  satisfying  $\alpha_C \geq \gamma > \alpha_C - \frac{1}{2}$ .

**Case 2:** If  $\alpha_C - \gamma < 0$ , then we have

$$\begin{aligned} |(\mathcal{C}z)(t)| &= \|C\|_{\mathcal{L}(Z_{\alpha_C}, \mathbb{C})} \|A^{\alpha_C - \gamma} T(t)\|_{\mathcal{L}(Z)} \|z\|_{Z_\gamma} \\ &\leq \|C\|_{\mathcal{L}(Z_{\alpha_C}, \mathbb{C})} K_2 e^{-\epsilon t} \|z\|_{Z_\gamma}, \end{aligned}$$

and  $t \mapsto e^{-\epsilon t} \in L_2([0, \infty), \mathbb{C})$ . So by Theorem 3.2.2, it follows that  $\mathcal{C}$  is Hilbert-Schmidt for all  $\gamma$  satisfying  $\alpha_C < \gamma$ .

From the above cases, it follows that  $\mathcal{C}$  is Hilbert-Schmidt for all  $\gamma$  satisfying  $\gamma > \alpha_C - \frac{1}{2}$ .

Since  $\mathcal{B}$  is an admissible control operator for  $\{T(t)\}_{t \geq 0}$  with state space  $Z_\gamma$ ,  $\mathcal{B}$  is an element in  $\mathcal{L}(L_2([0, \infty), \mathbb{C}^m), Z_\gamma)$  and for  $u \in L_2([0, \infty), \mathbb{C}^m)$ ,

$$Bu = \int_0^\infty T(t)Bu(t)dt.$$

Thus the dual operator  $B' \in \mathcal{L}(Z'_{-\gamma}, L_2([0, \infty), \mathbb{C}^m))$  and for  $z \in Z'_{-\gamma}$ ,

$$(B'z)(\cdot) = B'T(\cdot)'z.$$

Proceeding as above, it can be shown that  $B' \in \mathcal{L}(Z'_{-\gamma}, L_2([0, \infty), \mathbb{C}^m))$  is Hilbert-Schmidt. Thus, using Proposition 2 (page 261, Aubin [4]) it follows that  $B$  is also Hilbert-Schmidt.

3. If  $u \in \mathbb{C}^m$ , then  $Bu \in Z_{\alpha_B}$  and for  $t > 0$ ,  $T(t)Bu \in \cap_{\beta \in \mathbb{R}} Z_{\beta}$ . Consequently  $T(t)Bu \in Z_{\alpha_C}$  and  $CT(t)Bu \in \mathbb{C}^p$  for every  $t > 0$ . Moreover,

$$\|CT(t)Bu\|_{\mathbb{C}^p} \leq \|C\|_{\mathcal{L}(Z_{\alpha_C}, \mathbb{C}^p)} \|A^{\alpha_C - \alpha_B} T(t)\|_{\mathcal{L}(Z)} \|B\|_{\mathcal{L}(\mathbb{C}^m, Z_{\alpha_C})} \|u\|_{\mathbb{C}^m}.$$

But since  $\alpha_C - \alpha_B \geq 0$ , it follows that

$$\|CT(t)B\|_{\mathbb{C}^p \times \mathbb{R}^m} \leq \|C\|_{\mathcal{L}(Z_{\alpha_C}, \mathbb{C}^p)} K_1 \frac{e^{-\epsilon t}}{t^{\alpha_C - \alpha_B}} \|B\|_{\mathcal{L}(\mathbb{C}^m, Z_{\alpha_C})}.$$

Finally, since  $\alpha_C - \alpha_B < 1$ , we obtain that  $h(\cdot) = CT(\cdot)B \in L_1([0, \infty), \mathbb{C}^p \times \mathbb{R}^m)$ .

We know that  $B \in \mathcal{L}(L_2([0, \infty), \mathbb{C}^m), Z_{\gamma})$  and  $C \in \mathcal{L}(Z_{\gamma}, L_2([0, \infty), \mathbb{C}^p))$ . Consequently, if  $u \in L_2([0, \infty), \mathbb{C}^m)$ ,  $Bu \in Z_{\gamma}$ , and

$$(C(Bu))(t) = CT(t)Bu = CT(t) \int_0^{\infty} T(\tau)Bu(\tau)d\tau.$$

But  $CT(t) \in \mathcal{L}(Z_{\gamma}, \mathbb{C}^p)$  and so we have (see for example Theorem A.5.23, page 628, Curtain and Zwart [27])

$$\begin{aligned} (C(Bu))(t) &= \int_0^{\infty} CT(t)T(\tau)Bu(\tau)d\tau \\ &= \int_0^{\infty} CT(t+\tau)Bu(\tau)d\tau \\ &= \int_0^{\infty} h(t+\tau)u(\tau)d\tau \\ &= (\Gamma_h u)(t). \end{aligned}$$

Furthermore,  $B$  and  $C$  are Hilbert-Schmidt operators, and so it follows from Theorem 7.10 (b) (page 175, Weidmann [85]) that  $\Gamma_h = CB$  is nuclear.  $\blacksquare$

As explained in the introduction, analytic semigroups are generated by parabolic and some hyperbolic partial differential equations and so the above theorem is relevant to many distributed parameter systems with unbounded sensing and control. For Example 2.4.10 considered earlier in Section 2.4 of Chapter 2, the transfer function  $G(s) = \frac{1}{(1+s)^m}$  has a nuclear Hankel operator.

The analyticity assumption in Theorem 3.2.3 is crucial. In general, if  $A$  generates an exponentially stable semigroup and if  $B$  or  $C$  is unbounded, the

Hankel operator will not be nuclear: to show this, we revisit Example 2.3.4 given in Chapter 2.

**Example 2.3.4** (continued) We note that the semigroup is not analytic, but it is exponentially stable, and  $B$  is bounded, but  $C$  is unbounded.

The Hankel singular values are  $\sigma_k = \frac{1}{\sqrt{\mu_k^2 + 1}}$ , where the  $\mu_k$ 's are the roots of the transcendental equation

$$\tan(\mu\tau) = \frac{-\mu(3 - \mu^2)}{1 - 3\mu^2} \quad (3.9)$$

(see for example, Theorem 8.2.10, pages 402-403, Curtain and Zwart [27]).

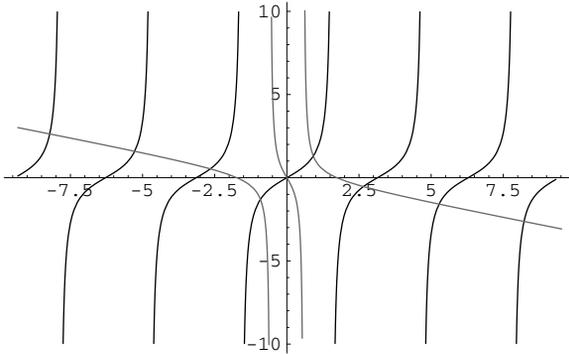


Figure 3.4: Roots of the transcendental equation.

For the sake of simplicity, we assume  $\tau = 1$ . Because of the periodicity of  $\tan(\cdot)$ , its monotonicity in each periodic interval of the type

$$\left( (2k-1)\frac{\pi}{2}, (2k+1)\frac{\pi}{2} \right), \quad k \in \mathbb{Z},$$

and the monotonicity of the function  $f(x) = \frac{-x(3-x^2)}{1-3x^2}$ , for  $|x| > \sqrt{3}$ , the positive roots of the transcendental equation above satisfy  $1 < \mu_k < k\pi$  for  $k > 2$  (see Figure 3.4). Thus we obtain

$$\sigma_k = \frac{1}{\sqrt{\mu_k^2 + 1}} > \frac{1}{\sqrt{\mu_k^2 + \mu_k^2}} = \frac{1}{\sqrt{2}\mu_k} > \frac{1}{\sqrt{2}\pi k} \quad \text{for } k > 2.$$

Hence  $\sum \sigma_k$  diverges, and so  $\Gamma_h$  is not nuclear. However, it is Hilbert-Schmidt. This follows from Exercise 8.9 (Curtain and Zwart [27]).  $\diamond$

Finally, we make the point that exponential stability is not a necessary condition for nuclearity: in the following example, the Hankel operator is nuclear, although the semigroup is not exponentially stable.

**Example 3.2.4** We construct the example below following Ober [58]. Let  $\{\lambda_k\}_{k \geq 1}$  be a decreasing bounded sequence of distinct positive numbers converging to 0. Consider the system  $\Sigma(A, B, B^*)$ , with

$$A = \left[ \frac{-\lambda_k \lambda_1}{k1(\lambda_k + \lambda_1)} \right]_{1 \leq k, 1 < \infty}$$

and

$$B = \begin{bmatrix} \frac{\lambda_1}{1} \\ \frac{\lambda_2}{2} \\ \frac{\lambda_3}{3} \\ \vdots \end{bmatrix} \in \mathcal{L}(\mathbb{C}, \ell_2(\mathbb{N})).$$

Then

1.  $A \in \mathcal{L}(\ell_2(\mathbb{N}))$  is Hilbert-Schmidt. This follows from the proof of Proposition 3.(i) (page 304, Ober [58]) and Theorem 6.22, Weidmann [85]). Thus  $0 \in \sigma(A)$  and this rules out exponential stability. However, from the proof of Proposition 3.(iii) (page 304, [58]), it follows that the semigroup  $\{e^{At}\}_{t \geq 0}$  is strongly stable.
2. The Lyapunov equation

$$A\Lambda + \Lambda A = -BB^*. \quad (3.10)$$

has a solution

$$\Lambda_0 := \begin{bmatrix} \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_2 & 0 & \dots \\ 0 & 0 & \lambda_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

This is proved in Proposition 3 (page 304, [58]), and so it follows from Theorem 3.1 (page 10, Hansen and Weiss [45]) that  $B$  is an infinite-time admissible control operator. Since  $A = A^*$  generates a strongly stable semigroup, it follows from [45] (page 10) that (3.10) has the unique solution

$$BB^* = \Lambda_0 = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_2 & 0 & \dots \\ 0 & 0 & \lambda_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

which is clearly compact. Hence it follows from Theorem 6.4.(c) (page 131, Weidmann [85]) that  $\mathcal{B}$  is compact.

3. The time domain Hankel operator<sup>1</sup>  $\Gamma_h = \mathcal{B}^* \mathcal{B} \in \mathcal{L}(L_2(0, \infty))$  is compact, with  $\sigma_k(\Gamma_h) = \lambda_k$  for all  $k \in \mathbb{N}$ .

<sup>1</sup>The equality  $\Gamma_h = \mathcal{B}^* \mathcal{B}$  follows from the remark after the proof of Corollary 4.4, page 276, Ober [59]. That  $\sigma_k(\Gamma_h) = \lambda_k$  is established in Corollary 2 (page 304, [58]).

Finally, upon choosing the sequence  $\{\lambda_k\}_{k \in \mathbb{N}}$  such that  $\sum_{k \in \mathbb{N}} \lambda_k < \infty$ , we can obtain nuclearity: for example, we can take  $\lambda_k = \frac{1}{k^2}$ .  $\diamond$

**Conclusions:** There already exist necessary and sufficient conditions for the compactness and nuclearity of the Hankel operator in terms of the transfer function. However, in most applications, the model is given in terms of a triple  $(A, B, C)$  of operators.

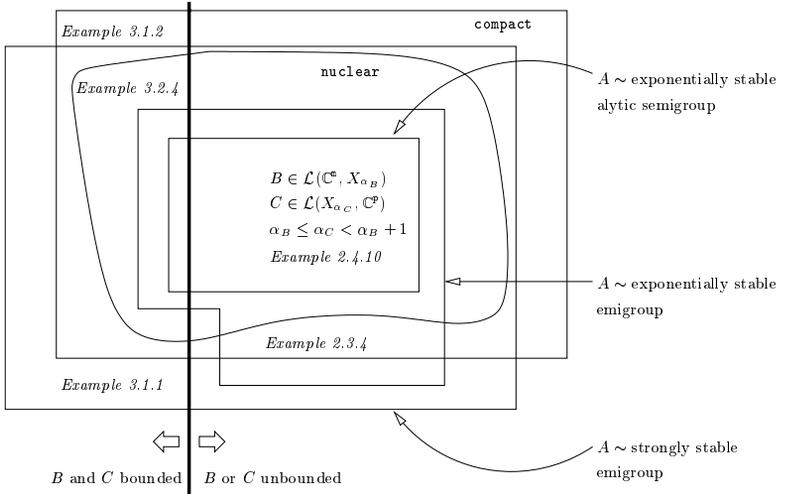


Figure 3.5: Overview of the results.

We have given new sufficient conditions for nuclearity for two classes of realizations: an exponentially stable realization with bounded  $B$  and  $C$  (Theorem 3.2.1) and an exponentially stable analytic realization (Theorem 3.2.3). On the other hand, by means of examples, we have shown that the state space properties of exponential and strong stability of the semigroup have little to do with the compactness and nuclearity of the Hankel operator. Figure 3.5 gives a concise overview of the results shown in the previous sections of this chapter.

### 3.3 $L_\infty$ -error of sub-optimal Hankel norm approximants

In this section, we prove that a sub-optimal Hankel norm approximant is also a good approximation with respect to the  $L_\infty$ -norm under the assumption that the Hankel operator is nuclear. To prove Theorem 3.3.1 below, we will use the following results:

- P1. If  $h \in L_1([0, \infty), \mathbb{C}^{\times m})$  and  $\Gamma_h$  is nuclear, then  $\|h\|_{L_1([0, \infty), \mathbb{C}^{\times m})} \leq 2 \sum_{k=1}^\infty \sigma_k$  (see for example, Theorem 2.1, page 866, Glover et al. [37]).
- P2. If  $K_1, K_2$  are compact operators from the Hilbert space  $\mathcal{H}_1$  to the Hilbert space  $\mathcal{H}_2$ , then  $\sigma_{j+k-1}(K_1 + K_2) \leq \sigma_j(K_1) + \sigma_k(K_2)$  for all  $j$  and  $k$  in  $\mathbb{N}$  (see for example, Theorem 7.7, page 171, Weidmann [85]).

**Theorem 3.3.1** *Suppose that*

1.  $h \in L_1([0, \infty), \mathbb{C}^{\times m})$ , and let  $G$  denote the Laplace transform of  $h$ ,
2.  $\Gamma_h$  is a nuclear Hankel operator,
3.  $\sigma_{1+1}(G) < \sigma < \sigma_1(G)$ , and
4.  $K(\cdot) \in H_{\infty, l}(\mathbb{C}^{\times m})$  is a solution to the sub-optimal Hankel norm approximation problem, that is,  $\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma$ .

If

- $K(\cdot) = G_*(\cdot) + F(\cdot)$ , where  
 $G_*(\cdot)$  is the rational transfer function of a finite-dimensional system with MacMillan degree at most 1, and  
 $F(\cdot) \in H_\infty(\mathbb{C}^{\times m})$ ,

then

$$\|G(i\cdot) + G_*(i\cdot)\|_\infty \leq 41 \sigma_1(G) + 2 \sum_{k=1+1}^\infty \sigma_k(G).$$

**Proof** We know that

$$\sigma_k(H_{G+G_*}) \leq \|H_{G+G_*}\| = \|H_{G+K}\| \leq \|G + K\|_\infty \leq \sigma < \sigma_1, \tag{3.11}$$

for all  $k \in \mathbb{N}$ . Moreover from P2. above, we obtain

$$\sigma_{k+1}(H_{G+G_*}) \leq \sigma_k(H_G) + \sigma_{1+1}(H_{G_*}) = \sigma_k(H_G), \tag{3.12}$$

since  $G_*$  has MacMillan degree at most equal to 1. Thus we have

$$\left. \begin{array}{l} \sigma_1(H_{G+G_*}) \leq \sigma_1(G) \\ \vdots \\ \sigma_1(H_{G+G_*}) \leq \sigma_1(G) \\ \sigma_{1+1}(H_{G+G_*}) \leq \sigma_1(G) \\ \vdots \\ \sigma_{1+1}(H_{G+G_*}) \leq \sigma_1(G) \\ \sigma_{1+1+1}(H_{G+G_*}) \leq \sigma_{1+1}(G) \\ \vdots \end{array} \right\} \begin{array}{l} \text{using (3.11)} \\ \\ \text{(3.11) and (3.12) both apply,} \\ \text{but (3.11) is sharper.} \\ \\ \text{(3.11) and (3.12) both apply,} \\ \text{but (3.12) is sharper.} \end{array}$$

Finally, employing the estimate from P1. above, we have

$$\|G(i\cdot) + G_*(i\cdot)\|_\infty \leq \|h + h_*\|_{L_1([0,\infty), \mathbb{C}^{\times n})} \leq 2 \sum_{k=1}^{\infty} \sigma_k(H_{G+G_*}),$$

where  $h_*$  denotes the inverse Laplace transform of  $G_*$ . Thus we have

$$\|G(i\cdot) + G_*(i\cdot)\|_\infty \leq 41 \sigma_1(G) + 2 \sum_{k=1+1}^{\infty} \sigma_k(G).$$

■

The error bound in Theorem 3.3.1 is like the one in Theorem 6.4 (page 889, Glover et al. [37]).

Finally we investigate the nature of the error bound

$$E_1 := 41 \sigma_1(G) + 2 \sum_{k=1+1}^{\infty} \sigma_k(G).$$

In Theorem 3.3.3 below, we prove that  $E_1 \rightarrow 0$  as  $1 \rightarrow \infty$ . This shows that the sub-optimal Hankel norm approximants are also good approximations with respect to the  $L_\infty$ -norm. In order to prove Theorem 3.3.3 below, we need the following lemma.

**Lemma 3.3.2** *If  $a_1 \geq a_2 \geq \dots \geq 0$  and  $\sum_{n=1}^{\infty} a_n < \infty$ , then  $\lim_{n \rightarrow \infty} n a_n = 0$ .*

**Proof** We have

$$\sum_{k=n+1}^{\infty} a_k \geq a_{n+1} + \dots + a_{2n} \geq n a_{2n} \geq 0.$$

Thus  $\lim_{n \rightarrow \infty} n a_{2n} = 0$ , and so

$$\lim_{n \rightarrow \infty} 2n a_{2n} = 0. \tag{3.13}$$

Furthermore,

$$\sum_{k=n+1}^{\infty} a_k \geq a_{n+2} + \dots + a_{2n+1} \geq n a_{2n+1} \geq 0.$$

Thus  $\lim_{n \rightarrow \infty} n a_{2n+1} = 0$ , and so  $\lim_{n \rightarrow \infty} 2n a_{2n+1} = 0$ . But since  $\lim_{n \rightarrow \infty} a_{2n+1} = 0$ , we obtain

$$\lim_{n \rightarrow \infty} (2n + 1) a_{2n+1} = 0. \tag{3.14}$$

From (3.13) and (3.14) above, the result follows. ■

**Remark:** We remark that the assumption  $a_1 \geq a_2 \geq \dots \geq 0$  is important. Consider for example the lacunary series in which the  $(n^2)^{\text{th}}$  term is  $\frac{1}{n^2}$ , and all other terms are zero: then

$$\frac{1}{1^2} + 0 + 0 + \frac{1}{2^2} + 0 + 0 + 0 + 0 + \frac{1}{3^2} + 0 + \dots < \infty,$$

but  $\{n a_n\}$  has the constant subsequence  $\{n^2 \frac{1}{n^2}\}$ .

**Theorem 3.3.3** *If  $H_G$  is nuclear, then given any  $\epsilon > 0$ , there exists a  $N \in \mathbb{N}$  such that*

$$E_1 = 41 \sigma_1(G) + 2 \sum_{k=1+1}^{\infty} \sigma_k(G) < \epsilon \text{ for all } 1 \geq N.$$

*That is,  $E_1 \rightarrow 0$  as  $1 \rightarrow \infty$ .*

**Proof** Since  $H_G$  is nuclear,  $\sum \sigma_k(G)$  converges, and so  $\sum_{k=1+1}^{\infty} \sigma_k(G) \rightarrow 0$ . From Lemma 3.3.2 above, it follows that  $\lim_{1 \rightarrow \infty} 1 \sigma_1(G) = 0$ . Hence the result follows. ■

# Chapter 4

## Characterization of all solutions

In this chapter, we characterize all solutions to the sub-optimal Hankel norm approximation problem in terms of a solution to a key  $J$ -spectral factorization problem. All the proofs in this chapter are based on purely “frequency domain” techniques.

In Chapter 5 we will give formulas for a  $J$ -spectral factor in terms of the state space parameters  $A$ ,  $B$  and  $C$  which satisfies the assumptions in this chapter, hence solving the sub-optimal Hankel norm approximation problem for the Pritchard-Salamon class and the analytic class of infinite-dimensional systems, respectively.

### 4.1 The sub-optimal Hankel norm approximation problem

Although it is known (see Adamjan et al. [2]) that if  $G(i\cdot) \in L_\infty(\mathbb{R}, \mathbb{C}^{p \times m})$  then

$$\inf_{K(\cdot) \in H_{\infty,1}(\mathbb{C}^{p \times n})} \|G(i\cdot) + K(i\cdot)\|_\infty = \sigma_{1+1},$$

in this section, we will prove by simpler methods that if  $G(i\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{C}^{p \times m})$  with limit 0 at  $\pm\infty$ , then

$$\inf_{K(\cdot) \in H_{\infty,1}(\mathbb{C}^{p \times n})} \|G(i\cdot) + K(i\cdot)\|_\infty = \sigma_{1+1}.$$

In order to do this, we will start by first proving the following simple result.

**Lemma 4.1.1** *Let  $S : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  be a compact, bounded operator from the Hilbert space  $\mathcal{H}_1$  to the Hilbert space  $\mathcal{H}_2$  with singular values  $\sigma_1 \geq \sigma_2 \geq \dots \geq 0$ . If  $S_* : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  is an arbitrary bounded linear map of rank 1, then  $\|S - S_*\| \geq \sigma_{1+1}$ .*

**Proof** If  $\sigma_{1+1} = 0$ , then the inequality is trivially satisfied. Let us assume that  $\sigma_{1+1} > 0$ . Suppose that  $\{w_k\}$  is an orthonormal sequence of eigenvectors of the operator  $SS^*$ :  $SS^*w_k = \sigma_k^2 w_k$ ,  $\|w_k\| = 1$ . Define  $v_k = \frac{1}{\sigma_k} S^* w_k$ . It is easy to verify that the  $v_k$ 's are orthonormal and  $Sv_k = \sigma_k w_k$ . Let  $\Pi_*$  be the orthogonal projection from  $\mathcal{H}_2$  onto  $\text{span}\{w_1, w_2, \dots, w_{1+1}\}$ ; then  $\|\Pi_*(S - S_*)\| \leq \|S - S_*\|$ . Consider the following restriction of  $\Pi_* S_*$ :

$$\Pi_* S_* : \text{span}\{v_1, v_2, \dots, v_{1+1}\} \rightarrow \text{span}\{w_1, w_2, \dots, w_{1+1}\}$$

which has rank at most 1, and hence there exists a  $z \in \ker(\Pi_* S_*)$ ,  $\|z\| = 1$ . If  $z = \sum_{k=1}^{1+1} a_k v_k$ , then we have  $\sum_{k=1}^{1+1} |a_k|^2 = 1$ , and

$$\begin{aligned} \Pi_* S z &= \sum_{k=1}^{1+1} a_k \sigma_k w_k, \\ \|S - S_*\|^2 &\geq \|\Pi_* S z - \Pi_* S_* z\|^2 = \|\Pi_* S z\|^2 \\ &= \sum_{k=1}^{1+1} \sigma_k^2 a_k^2 \geq \sigma_{1+1}^2 \sum_{k=1}^{1+1} a_k^2 = \sigma_{1+1}^2. \end{aligned}$$

Thus  $\|S - S_*\| \geq \sigma_{1+1}$ . ■

**Theorem 4.1.2** *Let  $G(i \cdot) \in L_\infty(\mathbb{R}, \mathbb{C}^{p \times m})$  be such that the Hankel operator with symbol  $G$ , namely  $H_G$ , is compact. Then*

$$\inf_{K(\cdot) \in H_{\infty, l}^c(\mathbb{C}^{p \times m})} \|G(i \cdot) + K(i \cdot)\|_\infty \geq \sigma_{l+1}.$$

**Proof** Let  $K(\cdot) = G_*(\cdot) + F(\cdot)$ , where  $G_*(-\cdot)$  has MacMillan degree  $\leq l$ , and  $F(\cdot) \in H_\infty^c(\mathbb{C}^{p \times m})$ . Thus,

$$\|G(i \cdot) + K(i \cdot)\|_\infty \geq \|G + G_* + F\|_H = \|G + G_*\|_H = \|H_G + H_{G_*}\| \geq \sigma_{l+1}(G).$$

The first equality follows easily from Lemma 8.1.2.c and Examples 8.1.3 (page 388, Curtain and Zwart [27]). The last inequality is a consequence of Lemma 4.1.1 above and the classical result that the rank of the Hankel matrix of the transfer function of a finite-dimensional system is equal to its MacMillan degree (see for instance Kalman et al. [48]). ■

**Theorem 4.1.3** *If  $G(i \cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{C}^{\mathfrak{p} \times \mathfrak{n}})$  with limit 0 at  $\pm i\infty$ , then*

$$\inf_{K(-) \in H_{\infty,1}^c(\mathbb{C}^{\mathfrak{p} \times \mathfrak{n}})} \|G(i \cdot) + K(i \cdot)\|_{\infty} \leq \sigma_{1+1}.$$

**Proof 1.** From Hartman's theorem (see for instance Corollary 4.10, page 46, Partington [62]), it follows that the Hankel operator corresponding to  $G$  is compact. Consequently, from Theorem 4.1.2 above, we have

$$\inf_{K(-) \in H_{\infty,1}^c(\mathbb{C}^{\mathfrak{p} \times \mathfrak{n}})} \|G(i \cdot) + K(i \cdot)\|_{\infty} \geq \sigma_{1+1}.$$

2. We now prove the reverse inequality. From Theorem A.6.11 (page 640, Curtain and Zwart [27]), we have the existence of an approximating sequence  $\{G_n\}_{n \in \mathbb{N}}$  of rational transfer matrices with no poles on the imaginary axis for  $G$ . Given any  $\epsilon > 0$ , choose  $N \in \mathbb{N}$  large enough such that for all  $n \geq N$ ,

$$\|G(i \cdot) - G_n(i \cdot)\|_{\infty} < \frac{\epsilon}{3}. \quad (4.1)$$

Let

$$G_n = {}^s G_n + {}^u G_n,$$

where  ${}^s G_n(\cdot) \in H_{\infty}^c(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$  and  ${}^u G_n(\cdot) \in H_{\infty}^c(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$ . From Lemma 8.1.2.c and Example 8.1.3 (page 388, Curtain and Zwart [27]), it follows that

$$\begin{aligned} \sigma_{1+1}({}^s G_n) &= \sigma_{1+1}(H^s G_n) \\ &= \sigma_{1+1}(H^s G_n + H^u G_n) \\ &= \sigma_{1+1}(H^s G_n + {}^u G_n) \\ &= \sigma_{1+1}(H G_n). \end{aligned}$$

We know that if  $K_1$  and  $K_2$  are compact operators from the Hilbert space  $\mathcal{H}_1$  to the Hilbert space  $\mathcal{H}_2$ , then  $\sigma_{j+k-1}(K_1 + K_2) \leq \sigma_j(K_1) + \sigma_k(K_2)$  for all  $j$  and  $k$  in  $\mathbb{N}$  (see for example, Theorem 7.7, page 171, Weidmann [85]). Applying this with

$$K_1 = H_G, \quad K_2 = H_{G_n - G}, \quad j = 1 + 1, \quad k = 1,$$

for  $n \geq N$  we have

$$\begin{aligned} \sigma_{1+1}({}^s G_n) &= \sigma_{1+1}(G_n) \\ &\leq \sigma_{1+1}(G) + \sigma_1(G_n - G) \\ &\leq \sigma_{1+1}(G) + \|G_n - G\|_{\infty} \quad (\text{from Lemma 8.1.2, page 388, [27]}) \\ &\leq \sigma_{1+1}(G) + \frac{\epsilon}{3}. \end{aligned} \quad (4.2)$$

In Glover [35] (Theorem 3.1), it was shown that given a stable rational matrix  ${}^s G_n$ , there exists a  $K_n(\cdot) \in H_{\infty,1}^c(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$  such that

$$\|{}^s G_n(i \cdot) + K_n(i \cdot)\|_{\infty} \leq \sigma_{1+1}({}^s G_n) + \frac{\epsilon}{3}. \quad (4.3)$$

We note that since  $K_n(\cdot) \in H_{\infty,1}^c(\mathbb{C}^{\mathbb{P}^{\times m}})$  and  ${}^u G_n(\cdot) \in H_{\infty}^c(\mathbb{C}^{\mathbb{P}^{\times m}})$ , it follows that  $K_n(\cdot) + {}^u G_n(\cdot) \in H_{\infty,1}^c(\mathbb{C}^{\mathbb{P}^{\times m}})$ . Furthermore for  $n \geq \mathbf{n}$ , we have

$$\begin{aligned} \|G(i \cdot) + K_n(i \cdot) - {}^u G_n(i \cdot)\|_{\infty} &= \|G(i \cdot) - G_n(i \cdot) + K_n(i \cdot) + G_n(i \cdot) - {}^u G_n(i \cdot)\|_{\infty} \\ &\leq \|G(i \cdot) - G_n(i \cdot)\|_{\infty} + \|{}^s G_n(i \cdot) + K_n(i \cdot)\|_{\infty} \\ &\leq \frac{\epsilon}{3} + \sigma_{1+1}({}^s G_n) + \frac{\epsilon}{3} \quad (\text{using (4.1) and (4.3)}) \\ &\leq \frac{\epsilon}{3} + \sigma_{1+1}(G) + \frac{\epsilon}{3} + \frac{\epsilon}{3} \quad (\text{using (4.2)}) \\ &= \sigma_{1+1}(G) + \epsilon. \end{aligned}$$

But since the choice of  $\epsilon > 0$  was arbitrary, the claim follows.  $\blacksquare$

From the results shown in Theorems 4.1.2 and 4.1.3, we have the following:

**Theorem 4.1.4** *If  $G(i \cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{C}^{\mathbb{P}^{\times m}})$  with limit 0 at  $\pm\infty$ , then*

$$K(-) \inf_{K(-) \in H_{\infty,1}^c(\mathbb{C}^{\mathbb{P}^{\times m}})} \|G(i \cdot) + K(i \cdot)\|_{\infty} = \sigma_{1+1}.$$

In light of the above theorem, we are now ready to state the *sub-optimal Hankel norm approximation problem* that we consider:

Let  $G(i \cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{C}^{\mathbb{P}^{\times m}})$  with limit 0 at  $\pm\infty$ .  
If  $\sigma_{1+1} < \sigma < \sigma_1$ , then find  $K(-) \in H_{\infty,1}^c(\mathbb{C}^{\mathbb{P}^{\times m}})$   
such that  $\|G(i \cdot) + K(i \cdot)\|_{\infty} \leq \sigma$ .

$K$  is then called a *solution* of the sub-optimal Hankel norm approximation problem.

In the following corollary, we show that any solution  $K_0(-) \in H_{\infty,1}^c(\mathbb{C}^{\mathbb{P}^{\times m}})$  (with *at most 1* unstable poles) of the sub-optimal Hankel norm approximation problem in fact has an unstable rational part of MacMillan degree *exactly 1*!

**Corollary 4.1.5** *If  $K(-) \in H_{\infty,1}^c(\mathbb{C}^{\mathbb{P}^{\times m}})$  is such that  $\|G(i \cdot) + K(i \cdot)\|_{\infty} \leq \sigma < \sigma_1$ , then  $K(\cdot) = G_*(\cdot) + F(\cdot)$ , where  $G_*(\cdot)$  has MacMillan degree 1, with all 1 poles in the open right half-plane and  $F(\cdot) \in H_{\infty}^c(\mathbb{C}^{\mathbb{P}^{\times m}})$ , that is,  $K(-) \in H_{\infty,[1]}^c(\mathbb{C}^{\mathbb{P}^{\times m}})$ .*

**Proof** Suppose that  $K(\cdot) = G_{*1}(\cdot) + F_1(\cdot)$ , where  $G_{*1}(\cdot)$  has MacMillan degree  $\mathbf{r}$  and all  $\mathbf{r}$  poles in the open right half-plane, and  $F_1(\cdot) \in H_{\infty}^c(\mathbb{C}^{\mathbb{P}^{\times m}})$ . Since  $K(-) \in H_{\infty,1}^c(\mathbb{C}^{\mathbb{P}^{\times m}})$ ,  $\mathbf{r} \leq 1$ . From Theorem 4.1.2, it follows that

$$\sigma_1 > \sigma \geq \|G(i \cdot) + K(i \cdot)\|_{\infty} \geq \inf_{K(-) \in H_{\infty,r}^c(\mathbb{C}^{\mathbb{P}^{\times m}})} \|G(i \cdot) + K(i \cdot)\|_{\infty} \geq \sigma_{\mathbf{r}+1}.$$

Thus  $\sigma_1 > \sigma_{\mathbf{r}+1}$ , which implies that  $1 < \mathbf{r} + 1$ , and so  $1 \leq \mathbf{r}$ . Hence  $\mathbf{r} = 1$ .  $\blacksquare$

## 4.2 Main assumptions and a few useful consequences

In this section, we state the assumptions S1-S5 under which we solve the sub-optimal Hankel norm approximation problem. We will also prove a few elementary consequences of these assumptions which will be used in the sequel.

We make the following assumptions:

S1.  $G(i\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{C}^{p \times m})$  with limit 0 at  $\pm i\infty$ .

S2.  $\sigma_{1+m} < \sigma < \sigma_1$ , where  $\sigma_k$ 's denote the Hankel singular values of  $G$ .

S3. There exists a  $\Lambda(\cdot) \in H_\infty^c(\mathbb{C}^{(p+m) \times (p+m)})$  such that for  $\omega \in \mathbb{R}$

$$\begin{aligned} & \begin{bmatrix} I_p & 0 \\ G(i\omega)^* & I_m \end{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix} \begin{bmatrix} I_p & G(i\omega) \\ 0 & I_m \end{bmatrix} \\ &= \Lambda(i\omega)^* \begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix} \Lambda(i\omega). \end{aligned} \quad (4.4)$$

S4.  $\Lambda(\cdot)$  is invertible as an element of  $H_\infty^c(\mathbb{C}^{(p+m) \times (p+m)})$ , that is, there exists a  $V(\cdot) \in H_\infty^c(\mathbb{C}^{(p+m) \times (p+m)})$  such that  $\Lambda(-s)V(-s) = I_{p+m}$  for all  $s \in \overline{\mathbb{C}_0^+}$ .

S5.  $\lim_{\omega \rightarrow \pm\infty} \Lambda(i\omega) = \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix}$ .

### Remarks:

1. From Hartman's theorem (see for example Corollary 4.10, page 46, Partington [62]) and S1, we have that the Hankel operator with symbol  $G$  is compact.
2. From S5 and Lemma 2.5.9, it follows that

$$\lim_{\substack{s \rightarrow \infty \\ s \in \overline{\mathbb{C}_0^+}}} \Lambda(-s) = \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix}.$$

3. We remark that S5 is only a normalization condition. All we actually need is the existence of a unique limit of  $\Lambda(i\cdot)$  at  $\pm\infty$ . Indeed, if there exists a  $\Lambda_0$  satisfying S3 and S4 such that  $\Lambda_0(i\cdot)$  has a limit, say  $\Lambda_\infty$  at  $\pm\infty$ , then

it can be easily seen that  $\Lambda$  defined by

$$\Lambda(s) = \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix} \Lambda_\infty^{-1} \Lambda_0(s) \text{ for all } s \in \overline{\mathbb{C}_0^-}.$$

satisfies S3, S4 and S5.

**Lemma 4.2.1** *If the assumptions S1-S5 hold, then*

1.  $\lim_{\substack{s \rightarrow \infty \\ s \in \mathbb{C}_0^+}} V(-s) = \begin{bmatrix} I_p & 0 \\ 0 & \frac{1}{\sigma} I_m \end{bmatrix}$ .
2.  $V_{22}(i\omega)$  is invertible for all  $\omega \in \mathbb{R}$ , and  $\|V_{22}(i\cdot)^{-1}V_{21}(i\cdot)\|_\infty < 1$ .
3. There exists a  $K_*(-\cdot) \in H_{\infty,1}^c(\mathbb{C}^{p \times m})$  such that  $\|G(i\cdot) + K_*(i\cdot)\|_\infty \leq \sigma$ .
4.  $V_{22}(-\cdot)^{-1} \in H_{\infty,1}^c(\mathbb{C}^{p \times p})$ .
5.  $K_0(-\cdot) := V_{12}(-\cdot)V_{22}(-\cdot)^{-1} \in H_{\infty,[1]}^c(\mathbb{C}^{p \times m})$  and it satisfies  $\|G(i\cdot) + K_0(i\cdot)\|_\infty \leq \sigma$ .
6.  $(V_{12}(-\cdot), V_{22}(-\cdot))$  is a right coprime factorization over  $\mathcal{MH}_\infty^c$  of  $K_0(-\cdot) \in H_{\infty,[1]}^c(\mathbb{C}^{p \times m})$ .
7.  $\det(V_{22}(-\cdot))$  has no zeros on the imaginary axis, and exactly 1 zeros in  $\mathbb{C}_0^+$ .

**Proof 1.** From S5,

$$V(-s) - \begin{bmatrix} I_p & 0 \\ 0 & \frac{1}{\sigma} I_m \end{bmatrix} = V(-s) \left( \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix} - \Lambda(-s) \right) \begin{bmatrix} I_p & 0 \\ 0 & \frac{1}{\sigma} I_m \end{bmatrix},$$

and the fact that  $V(-\cdot) \in \mathcal{MH}_\infty^c$ , the result follows.

2.  $\Lambda$  satisfies S3, and so taking inverses, we obtain

$$\begin{aligned} & V(i\omega) \begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix} V(i\omega)^* \\ &= \begin{bmatrix} I_p & G(i\omega) \\ 0 & -I_m \end{bmatrix}^{-1} \begin{bmatrix} I_p & 0 \\ 0 & -\frac{1}{\sigma^2} I_m \end{bmatrix} \begin{bmatrix} I_p & 0 \\ G(i\omega)^* & -I_m \end{bmatrix}^{-1}. \end{aligned} \quad (4.5)$$

for all  $\omega \in \mathbb{R}$ . Considering the (2, 2)-block of the above yields

$$V_{21}(i\omega)V_{21}(i\omega)^* - V_{22}(i\omega)V_{22}(i\omega)^* = -\frac{1}{\sigma^2}I_m, \text{ where } \omega \in \mathbb{R}. \quad (4.6)$$

Thus for  $u \in \mathbb{C}^m$  we have

$$\|V_{22}(i\omega)^*u\|^2 = \|V_{21}(i\omega)^*u\|^2 + \frac{1}{\sigma^2}\|u\|^2.$$

Thus if  $V_{22}(i\omega)^*u = 0$ , then  $u = 0$ . So it follows that  $V_{22}(i\omega)^*$  is invertible for all  $\omega \in \mathbb{R}$ , or equivalently,  $V_{22}(i\omega)$  is invertible for all  $\omega \in \mathbb{R}$ .

From (4.6), we have  $\|V_{22}(i\omega)^{-1}V_{21}(i\omega)u\|^2 - \|u\|^2 = -\frac{1}{\sigma^2}\|V_{22}(i\omega)^{-1}u\|^2$ . Let  $M > 0$  be such that  $\|V_{22}(i\omega)\| \leq M$  for all  $\omega \in \mathbb{R}$ , we obtain  $\|u\|^2 \leq \|V_{22}(i\omega)\|^2 \|V_{22}(i\omega)^{-1}u\|^2 \leq M^2 \|V_{22}(i\omega)^{-1}u\|^2$ . Hence

$$\|V_{22}(i\omega)^{-1}V_{21}(i\omega)\|^2 \leq 1 - \frac{1}{\sigma^2 M^2} < 1 \text{ for all } \omega \in \mathbb{R},$$

and so we have  $\|V_{22}(i\cdot)^{-1}V_{21}(i\cdot)\|_\infty < 1$ .

3. From S3 it  $G(i\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{C}^{p \times m})$  and  $\lim_{\omega \rightarrow \pm\infty} G(i\omega) = 0$ . So from Theorem 4.1.4, the result follows.

4. From (1), we know that

$$\lim_{\substack{s \rightarrow \frac{\infty}{\sigma} \\ s \in \mathbb{C}_0^+}} V_{22}(-s) = \frac{1}{\sigma} I_m.$$

Thus applying Lemma 2.5.10 to  $V_{22}(-\cdot)$ , we obtain that  $V_{22}(-\cdot)^{-1} \in H_{\infty, \bullet}^c(\mathbb{C}^{k \times k})$ .

Let  $K_*(-\cdot) \in H_{\infty, [1]}^c(\mathbb{C}^{p \times m})$  satisfy  $\|G(i\cdot) + K_*(i\cdot)\|_\infty \leq \sigma$  and suppose it has the coprime factorization  $K_*(-\cdot) = NM^{-1}$  over  $\mathcal{M}H_\infty^c$ , where  $N$  and  $M$  are in  $\mathcal{M}H_\infty^c$ ,  $M$  is rational, and  $\det(M) \in \mathcal{R}_\infty$  has exactly 1 zeros in  $\mathbb{C}_0^+$  and none on the imaginary axis. Define

$$\begin{aligned} \begin{bmatrix} U_1(-\cdot) \\ U_2(-\cdot) \end{bmatrix} &= \begin{bmatrix} \Lambda_{11}(-\cdot)N(\cdot) + \Lambda_{12}(-\cdot)M(\cdot) \\ \Lambda_{21}(-\cdot)N(\cdot) + \Lambda_{22}(-\cdot)M(\cdot) \end{bmatrix} = \Lambda(-\cdot) \begin{bmatrix} N(\cdot) \\ M(\cdot) \end{bmatrix} \\ &= \Lambda(-\cdot) \begin{bmatrix} K_*(-\cdot) \\ I_m \end{bmatrix} M(\cdot). \end{aligned} \quad (4.7)$$

We have

$$\begin{bmatrix} U_1(i\omega) \\ U_2(i\omega) \end{bmatrix} = \Lambda(i\omega) \begin{bmatrix} I_p & G(i\omega) \\ 0 & I_m \end{bmatrix}^{-1} \begin{bmatrix} G(i\omega) + K_*(i\omega) \\ I_m \end{bmatrix} M(-i\omega), \quad \omega \in \mathbb{R}.$$

For  $\omega \in \mathbb{R}$ , note that

$$U_1(i\omega)^*U_1(i\omega) - U_2(i\omega)^*U_2(i\omega) = \begin{bmatrix} U_1(i\omega) \\ U_2(i\omega) \end{bmatrix}^* \begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix} \begin{bmatrix} U_1(i\omega) \\ U_2(i\omega) \end{bmatrix},$$

and appealing to S3, that

$$\begin{bmatrix} I_p & 0 \\ G(i\omega)^* & I_m \end{bmatrix}^{-1} \Lambda(i\omega)^* \begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix} \Lambda(i\omega) \begin{bmatrix} I_p & G(i\omega) \\ 0 & I_m \end{bmatrix}^{-1} = \begin{bmatrix} I_p & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix}.$$

Thus

$$\begin{aligned} & U_1(i\omega)^* U_1(i\omega) - U_2(i\omega)^* U_2(i\omega) \\ &= M(-i\omega)^* \begin{bmatrix} G(i\omega) + K_*(i\omega) \\ I_m \end{bmatrix}^* \begin{bmatrix} I_p & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix} \begin{bmatrix} G(i\omega) + K_*(i\omega) \\ I_m \end{bmatrix} M(-i\omega). \end{aligned} \quad (4.8)$$

Hence for all  $u \in \mathbb{C}^m$  and all  $\omega \in \mathbb{R}$ , we have from equation (4.8) that

$$\|U_1(i\omega)u\|^2 - \|U_2(i\omega)u\|^2 = \|(G(i\omega) + K_*(i\omega))M(-i\omega)u\|^2 - \sigma^2 \|M(-i\omega)u\|^2 \leq 0, \quad (4.9)$$

since  $\|G(i\cdot) + K_*(i\cdot)\|_\infty \leq \sigma$ . So

$$\|U_1(i\omega)u\| \leq \|U_2(i\omega)u\|. \quad (4.10)$$

From (4.7) we obtain

$$V(\cdot) \begin{bmatrix} U_1(\cdot) \\ U_2(\cdot) \end{bmatrix} = \begin{bmatrix} K_*(\cdot) \\ I_m \end{bmatrix} M(\cdot), \quad (4.11)$$

and so

$$V_{21}(\cdot)U_1(\cdot) + V_{22}(\cdot)U_2(\cdot) = M(\cdot). \quad (4.12)$$

We claim that  $\ker(U_2(-i\omega)) = \{0\}$  for all  $\omega \in \mathbb{R}$ . Suppose on the contrary that there exists  $0 \neq u_0 \in \mathbb{C}^m$  and a  $\omega_0 \in \mathbb{R}$  such that  $U_2(-i\omega_0)u_0 = 0$ . Then from (4.10) and (4.12), we obtain  $M(-i\omega_0)u_0 = 0$ , which implies that  $u_0 = 0$ , a contradiction.

From (4.9), we deduce that

$$\|U_1(i\omega)U_2(i\omega)^{-1}y\|^2 \leq \|y\|^2 \text{ for all } \omega \in \mathbb{R},$$

and so  $U_1(i\cdot)U_2(i\cdot)^{-1} \in L_\infty(\mathbb{R}, \mathbb{C}^{p \times m})$  satisfies  $\|U_1(i\cdot)U_2(i\cdot)^{-1}\|_\infty \leq 1$ .

Consider  $U_2(\cdot) \in \mathcal{MH}_\infty^c$ . We know that  $\Lambda_{21}(\cdot)$  is strictly proper and both  $\Lambda_{22}(\cdot)$  and  $M(\cdot)$  are proper with invertible limits at infinity in  $\overline{\mathbb{C}}_0^+$ . Thus from (4.7) we see that

$$\lim_{\substack{s \rightarrow \infty \\ s \in \overline{\mathbb{C}}_0^+}} U_2(-s) \text{ exists and is invertible.} \quad (4.13)$$

Thus it follows from the proof of Lemma 2.5.10 that  $s \mapsto \det(U_2(-s))$  has only finitely many zeros in  $\overline{\mathbb{C}}_0^+$ , and they are all contained in  $\mathbb{C}_0^+$ .

The zeros of  $\det(V_{22}(\cdot))$ ,  $\det(M(\cdot))$  and  $\det(U_2(\cdot))$  are contained in some half-plane  $\mathbb{C}_\epsilon^+$ , where  $\epsilon > 0$ . Since  $\|V_{22}(i\cdot)^{-1}V_{21}(i\cdot)\|_\infty < 1$ , there exists a  $0 < r < 1$  such that  $\|V_{22}(i\cdot)^{-1}V_{21}(i\cdot)\|_\infty = 1 - r$ . It follows from Lemma 2.5.7 that there exists a  $\delta_1 > 0$  such that  $\delta_1 < \epsilon$  and for any  $\zeta$  satisfying  $0 < \zeta < \delta_1$ ,  $\|V_{22}(-\zeta - i\cdot)^{-1}V_{21}(-\zeta - i\cdot)\|_\infty \leq 1 - \frac{r}{2}$ . Similarly it follows from Lemma 2.5.7 that there exists a  $\delta_2 > 0$  such that  $\delta_2 < \epsilon$  and for any  $\zeta$  satisfying  $0 < \zeta < \delta_2$ ,

$$\|U_1(-\zeta - i\cdot)U_2(-\zeta - i\cdot)\|_\infty \leq 1 + \frac{\frac{r}{4}}{1 - \frac{r}{4}} = \frac{1}{1 - \frac{r}{4}}.$$

Let  $\delta := \min\{\delta_1, \delta_2\}$ , and fix a  $\zeta$  satisfying  $0 < \zeta < \delta$ . Define

$$\phi(\alpha, s) = \det(\alpha V_{21}(-\zeta - s)U_1(-\zeta - s) + V_{22}(-\zeta - s)U_2(-\zeta - s)),$$

where  $\alpha \in [0, 1]$ .

a. We know that

$$\begin{aligned}\phi(0, \cdot) &= \det(V_{22}(-\zeta - \cdot)U_2(-\zeta - \cdot)) \quad \text{and} \\ \phi(1, \cdot) &= \det(V_{21}(-\zeta - \cdot)U_1(-\zeta - \cdot) + V_{22}(-\zeta - \cdot)U_2(-\zeta - \cdot))\end{aligned}$$

are meromorphic (in fact analytic!) in  $\mathbb{C}_{-\zeta/2}^+$ .

b.  $\phi(0, \cdot)$  has a nonzero limit at infinity in  $\overline{\mathbb{C}_0^+}$ :  $\det(V_{22}(\cdot))$  has a nonzero limit at infinity in  $\overline{\mathbb{C}_0^+}$  and  $\det(U_2(\cdot))$  has a nonzero limit at infinity in  $\overline{\mathbb{C}_0^+}$  (see (4.13)).

$\phi(1, \cdot)$  has a nonzero limit at infinity in  $\overline{\mathbb{C}_0^+}$ , since  $V_{21}(\cdot)$  is strictly proper,  $U_1(\cdot)$  is proper in  $\overline{\mathbb{C}_0^+}$ , and the above.

c.  $(\alpha, s) \mapsto \phi(\alpha, s) : [0, 1] \times i\mathbb{R} \rightarrow \mathbb{C}$  is a continuous function, and

$$\begin{aligned}\phi(0, i\omega) &= \det(V_{22}(-\zeta - i\omega)U_2(-\zeta - i\omega)) \\ &= \det(V_{22}(-\zeta - i\omega)) \det(U_2(-\zeta - i\omega)), \quad \text{and} \\ \phi(1, i\omega) &= \det(V_{21}(-\zeta - i\omega)U_1(-\zeta - i\omega) + V_{22}(-\zeta - i\omega)U_2(-\zeta - i\omega)).\end{aligned}$$

d. We have

$$\begin{aligned}\phi(\alpha, i\omega) &= \det(V_{22}(-\zeta - i\omega)) \det(U_2(-\zeta - i\omega)) \\ &\quad \det(I + \alpha V_{22}(-\zeta - i\omega)^{-1}V_{21}(-\zeta - i\omega)U_1(-\zeta - i\omega)U_2(-\zeta - i\omega)^{-1}) \\ &\neq 0,\end{aligned}$$

since

$$\begin{aligned}&\|\alpha V_{22}(-\zeta - i\cdot)^{-1}V_{21}(-\zeta - i\cdot)U_1(-\zeta - i\cdot)U_2(-\zeta - i\cdot)^{-1}\|_\infty \\ &\leq 1 \|V_{22}(-\zeta - i\cdot)^{-1}V_{21}(-\zeta - i\cdot)\|_\infty \|U_1(-\zeta - i\cdot)U_2(-\zeta - i\cdot)^{-1}\|_\infty \\ &\leq \left[1 - \frac{r}{2}\right] \frac{1}{1 - \frac{r}{4}} < 1,\end{aligned}$$

$\det(V_{22}(-\zeta - i\omega)) \neq 0$  and  $\det(U_2(-\zeta - i\omega)) \neq 0$ .

*e.*  $\phi(\alpha, \infty) \neq 0$ , since  $V_{21}(\cdot)$  is strictly proper,  $U_1(\cdot)$  is proper in  $\overline{\mathbb{C}_0^+}$ , and  $\det(V_{22}(\cdot)) \det(U_2(\cdot))$  has a nonzero limit at infinity in  $\overline{\mathbb{C}_0^+}$ .

Thus the assumptions in Lemma A.1.18 (Curtain and Zwart [27], page 570) are satisfied by  $\phi$ , and hence it follows that the Nyquist indices of  $\phi(0, \cdot)$  and  $\phi(1, \cdot)$  are the same. Consequently, the number of zeros are the same (the number of poles is zero, as  $\phi(0, \cdot)$ ,  $\phi(1, \cdot)$  are analytic in  $\mathbb{C}_{-\frac{\delta}{2}}^+$ ) and so the sum of the number of zeros of  $s \mapsto \det(V_{22}(-\zeta - s))$  in  $\overline{\mathbb{C}_0^+}$  plus the number of zeros of  $s \mapsto \det(U_2(-\zeta - s))$  in  $\overline{\mathbb{C}_0^+}$  equals the number of zeros of  $s \mapsto \det(V_{21}(\underline{-\zeta} - s)U_1(-\zeta - s) + V_{22}(-\zeta - s)U_2(-\zeta - s))$  ( $= \det(M(\zeta + s))$ , using 4.12) in  $\overline{\mathbb{C}_0^+}$ .

In particular, we obtain that the number of zeros of  $s \mapsto \det(V_{22}(-\zeta - s))$  in  $\overline{\mathbb{C}_0^+}$  is less than or equal to 1. But since the choice of  $\zeta$  can be made arbitrarily small, it follows that  $s \mapsto \det(V_{22}(-s))$  has at most 1 zeros in  $\overline{\mathbb{C}_0^+}$ . Thus  $V_{22}(\cdot) \in H_{\infty,1}^c(\mathbb{C}^{m \times m})$ .

5. Define  $K_0(\cdot) := V_{12}(\cdot)V_{22}(\cdot)^{-1}$ . From Lemma 2.5.5 and part 4 above, it follows that  $K_0(\cdot)$  is an element in  $H_{\infty,1}^c(\mathbb{C}^{p \times m})$ . We have

$$\begin{aligned} \begin{bmatrix} G(i\omega) + K_0(i\omega) \\ I_m \end{bmatrix} &= \begin{bmatrix} I_p & G(i\omega) \\ 0 & I_m \end{bmatrix} \begin{bmatrix} K_0(i\omega) \\ I_m \end{bmatrix} \\ &= \begin{bmatrix} I_p & G(i\omega) \\ 0 & I_m \end{bmatrix} V(i\omega) \begin{bmatrix} 0 \\ V_{22}(i\omega)^{-1} \end{bmatrix}, \end{aligned}$$

with  $\omega \in \mathbb{R}$ , and so we have (for notational convenience, we restrict writing out the argument  $i\omega$ )

$$\begin{aligned} &(G + K_0)^*(G + K_0) - \sigma^2 I_m \\ &= \begin{bmatrix} G + K_0 \\ I_m \end{bmatrix}^* \begin{bmatrix} I_p & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix} \begin{bmatrix} G + K_0 \\ I_m \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ V_{22}^{-1} \end{bmatrix}^* V^* \begin{bmatrix} I_p & G \\ 0 & I_m \end{bmatrix}^* \begin{bmatrix} I_p & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix} \begin{bmatrix} I_p & G \\ 0 & I_m \end{bmatrix} V \begin{bmatrix} 0 \\ V_{22}^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ V_{22}^{-1} \end{bmatrix}^* \begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix} \begin{bmatrix} 0 \\ V_{22}^{-1} \end{bmatrix}, \end{aligned}$$

where we have used S3 and the definition of  $V$ . Thus it follows that  $\|(G(i\omega) + K_0(i\omega))u\|^2 - \sigma^2 \|u\|^2 = -\|V_{22}(i\omega)^{-1}u\|^2$  for all  $u \in \mathbb{C}^m$  and all  $\omega \in \mathbb{R}$ . Hence it follows that  $\|G(i \cdot) + K_0(i \cdot)\|_{\infty} \leq \sigma$ .

Thus from Corollary 4.1.5, it follows that  $K_0(\cdot) \in H_{\infty,[1]}^c(\mathbb{C}^{p \times m})$ .

6. We now prove that  $(V_{12}(-\cdot), V_{22}(-\cdot))$  is a right coprime factorization over  $\mathcal{MH}_\infty^c$  of  $K_0(-\cdot) \in H_{\infty, [1]}^c(\mathbb{C}^{\mathbb{P} \times \mathbb{m}})$ :  $V_{12}(-\cdot)$  and  $V_{22}(-\cdot)$  belong to  $\mathcal{MH}_\infty^c$ . Moreover,

$$\Lambda_{22}(-s)V_{22}(-s) - (-\Lambda_{21}(-s))V_{12}(-s) = I,$$

where  $\Lambda_{22}(-\cdot)$  and  $-\Lambda_{21}(-\cdot)$  belong to  $\mathcal{MH}_\infty^c$ . Hence it follows that  $(V_{12}(-\cdot), V_{22}(-\cdot))$  is a right coprime factorization of  $V_{12}(-\cdot)V_{22}(-\cdot)^{-1}$ .

7. From Lemma 2.5.6, it follows that  $\det(V_{22}(-\cdot))$  has no zeros on the imaginary axis, and exactly 1 zeros in  $\mathbb{C}_0^+$ .  $\blacksquare$

### 4.3 All solutions to the sub-optimal Hankel norm approximation problem

In this section we obtain a nice parameterization of all solutions to the sub-optimal Hankel norm approximation problem under the assumptions S1-S5 listed in the previous section.

**Theorem 4.3.1** *Suppose that S1-S5 hold. If  $Q(-\cdot) \in H_\infty^c(\mathbb{C}^{\mathbb{P} \times \mathbb{m}})$  satisfies  $\|Q(i\cdot)\|_\infty \leq 1$ , and  $K(-\cdot) := R_1(-\cdot)R_2(-\cdot)^{-1}$ , where*

$$\begin{bmatrix} R_1(-\cdot) \\ R_2(-\cdot) \end{bmatrix} := \Lambda(-\cdot)^{-1} \begin{bmatrix} Q(-\cdot) \\ I_m \end{bmatrix}, \quad (4.14)$$

*then  $K(-\cdot) \in H_{\infty, [1]}^c(\mathbb{C}^{\mathbb{P} \times \mathbb{m}})$  and  $\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma$ .*

**Proof** *Step 1: We show that  $\det(V_{21}(-\cdot)Q(-\cdot) + V_{22}(-\cdot))$  has exactly 1 zeros in  $\mathbb{C}_0^+$ , and*

*they are contained in the open right half-plane.*

By Lemma 4.2.1.7, we know that  $s \mapsto \det(V_{22}(-s))$  has no zeros on the imaginary axis, and exactly 1 zeros in  $\mathbb{C}_0^+$ . So there exists an  $\epsilon > 0$  such that all its zeros are contained in the half-plane  $\mathbb{C}_\epsilon^+$ .

From Lemma 2.5.5, it follows that  $V_{22}(-\cdot)^{-1}V_{21}(-\cdot) \in H_{\infty, 1}^c(\mathbb{C}^{\mathbb{P} \times \mathbb{m}})$ . Thus, we have from Lemma 4.2.1.2 and Lemma 2.5.7 applied to  $V_{22}(-\cdot)^{-1}V_{21}(-\cdot)$ , that there exists a  $\delta$ ,  $0 < \delta < \epsilon$ , such that whenever  $0 \leq \zeta \leq \delta$ ,

$$\sup_{\omega \in \mathbb{R}} \|V_{22}(-\zeta - i\omega)^{-1}V_{21}(-\zeta - i\omega)\| < 1.$$

Fix such a  $\zeta > 0$ . Consider

$$\phi(\alpha, s) := \det(\alpha V_{21}(-\zeta - s)Q(-\zeta - s) + V_{22}(-\zeta - s)),$$

where  $\alpha \in [0, 1]$ . The maps  $\phi(0, \cdot)$  and  $\phi(1, \cdot)$  are meromorphic (actually analytic in  $\mathbb{C}_{-\frac{\zeta}{2}}^+$ ) on an open set containing  $\overline{\mathbb{C}_0^+}$  with nonzero limits at infinity in  $\overline{\mathbb{C}_0^+}$ . We have:

1.  $(\alpha, s) \mapsto \phi(\alpha, s) : [0, 1] \times i\mathbb{R} \rightarrow \mathbb{C}$  is a continuous function.
2.  $\phi(0, i\omega) = \det(V_{22}(-\zeta - i\omega))$ , and  
 $\phi(1, i\omega) = \det(V_{21}(-\zeta - i\omega)Q(-\zeta - i\omega) + V_{22}(-\zeta - i\omega))$ .
3.  $I_m + \alpha V_{22}(-\zeta - i\omega)^{-1}V_{21}(-\zeta - i\omega)Q(-\zeta - i\omega)$  is invertible, since

$$\begin{aligned} & \left\| \alpha V_{22}(-\zeta - i\omega)^{-1}V_{21}(-\zeta - i\omega)Q(-\zeta - i\omega) \right\| \\ & \leq \alpha \left\| V_{22}(-\zeta - i\omega)^{-1}V_{21}(-\zeta - i\omega) \right\| \left\| Q(-\zeta - i\omega) \right\| \\ & < 1. \end{aligned}$$

Moreover, since  $s \mapsto \det(V_{22}(-\zeta - s))$  has no zeros on the imaginary axis, it follows that  $\phi(\alpha, i\omega)$  is nonzero for all  $\alpha \in [0, 1]$  and  $\omega \in \mathbb{R}$ .

4.  $\phi(\alpha, \infty)$  is nonzero for all  $\alpha \in [0, 1]$ , since

$$\lim_{\substack{s \rightarrow \infty \\ s \in \mathbb{C}_0^+}} \det(\alpha V_{21}(-s - \zeta)Q(-s - \zeta) + V_{22}(-s - \zeta)) = \frac{1}{\sigma^m} \neq 0.$$

So the assumptions of Lemma A.1.18 (Curtain and Zwart [27], page 570) are satisfied by  $\phi$  and so the Nyquist indices of  $\phi(0, \cdot)$  and  $\phi(1, \cdot)$  are the same. Consequently, the number of zeros are the same (the number of poles in each case is zero, since  $\phi(0, \cdot)$ ,  $\phi(1, \cdot)$  are analytic in  $\mathbb{C}_{-\frac{\zeta}{2}}^+$ ). We already know from that when  $\alpha = 0$ ,  $\det(V_{22}(-\cdot - \zeta))$  has 1 zeros in the closed right half-plane  $\overline{\mathbb{C}_0^+}$ . Thus the number of zeros of  $\phi(1, \cdot)$  is 1. But since  $\zeta$  can be chosen arbitrarily small,  $\det(V_{21}(-\cdot)Q(-\cdot) + V_{22}(-\cdot))$  has exactly 1 zeros in  $\overline{\mathbb{C}_0^+}$ .

Finally,  $\det(V_{21}(-\cdot)Q(-\cdot) + V_{22}(-\cdot))$  has no zeros on the imaginary axis, since  $\det(I + V_{22}(-i\omega)^{-1}V_{21}(-i\omega)Q(-i\omega)) \neq 0$  for all  $\omega \in \mathbb{R}$ .

*Step 2: We show that  $K(-\cdot) := (V_{11}(-\cdot)Q(-\cdot) + V_{12}(-\cdot))(V_{21}(-\cdot)Q(-\cdot) + V_{22}(-\cdot))^{-1} \in H_{\infty,1}^c(\mathbb{C}^{\mathbb{P} \times \mathbb{m}})$ .*

$\det(V_{21}(-\cdot)Q(-\cdot) + V_{22}(-\cdot))$  has a nonzero limit at infinity in  $\overline{\mathbb{C}_0^+}$ , since  $Q(-\cdot) \in \mathcal{MH}_\infty^c$  is proper,  $V_{21}(-\cdot)$  is strictly proper,  $V_{22}(-\cdot)$  is proper in  $\overline{\mathbb{C}_0^+}$  and  $\det(V_{22}(-\cdot))$  has a nonzero limit at infinity in  $\overline{\mathbb{C}_0^+}$ . So by Lemma 2.5.4 and using Step 1 above, it follows that  $K(-\cdot) = (V_{11}(-\cdot)Q(-\cdot) + V_{12}(-\cdot))(V_{21}(-\cdot)Q(-\cdot) + V_{22}(-\cdot))^{-1}$  is a well-defined element of  $H_{\infty,1}^c(\mathbb{C}^{\mathbb{P} \times \mathbb{m}})$ .

*Step 3: We show that  $\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma$ .*

We know that  $\det(V_{22}(-\cdot))$  has no zeros on the imaginary axis and  $\|V_{22}(i\omega)^{-1}V_{21}(i\omega)Q(i\omega)\| < 1$  for all  $\omega \in \mathbb{R}$ . Thus  $R_2(i\omega) = V_{21}(i\omega)Q(i\omega) + V_{22}(i\omega)$  is invertible for every  $\omega \in \mathbb{R}$ . Consequently with  $s = i\omega$ , for all  $\omega \in \mathbb{R}$ ,

$$\begin{aligned} & (G + K)^*(G + K) - \sigma^2 I_m \\ &= (R_2^{-1})^* \begin{bmatrix} Q^* & I_m \end{bmatrix} V^* \begin{bmatrix} I_p & G \\ 0 & I_m \end{bmatrix}^* \begin{bmatrix} I_p & 0 \\ 0 & -\sigma^2 I_m \end{bmatrix} \begin{bmatrix} I_p & G \\ 0 & I_m \end{bmatrix} V \begin{bmatrix} Q^* \\ I_m \end{bmatrix} R_2^{-1} \\ &= (R_2^{-1})^* (Q^*Q - I_m) R_2^{-1}, \end{aligned}$$

where we have used S3. Thus  $\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma$ .

*Step 4: We show that  $K(-\cdot) \in H_{\infty, [1]}^c(\mathbb{C}^{\mathbb{P}^{\times m}})$ .*

Finally, from Corollary 4.1.5, and Steps 2 and 3 above, it follows that  $K(-\cdot) \in H_{\infty, [1]}^c(\mathbb{C}^{\mathbb{P}^{\times m}})$ .  $\blacksquare$

Next we show that any solution  $K(-\cdot) \in H_{\infty, [1]}^c(\mathbb{C}^{\mathbb{P}^{\times m}})$  to the sub-optimal Hankel norm approximation problem has the form (4.14).

**Theorem 4.3.2** *Suppose that S1-S5 hold. If  $K(-\cdot) \in H_{\infty, [1]}^c(\mathbb{C}^{\mathbb{P}^{\times m}})$  and  $\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma$ , then  $K(-\cdot) = R_1(-\cdot)R_2(-\cdot)^{-1}$ , where*

$$\begin{bmatrix} R_1(-\cdot) \\ R_2(-\cdot) \end{bmatrix} = \Lambda(-\cdot)^{-1} \begin{bmatrix} Q(-\cdot) \\ I_m \end{bmatrix}$$

for some  $Q(-\cdot) \in H_\infty^c(\mathbb{C}^{\mathbb{P}^{\times m}})$  satisfying  $\|Q(i\cdot)\|_\infty \leq 1$ .

**Proof** Let  $K_0(-\cdot) \in H_{\infty, [1]}^c(\mathbb{C}^{\mathbb{P}^{\times m}})$  satisfy  $\|G(i\cdot) + K_0(i\cdot)\|_\infty \leq \sigma$  and suppose it has the coprime factorization  $K_0(-\cdot) = NM^{-1}$  over  $\mathcal{MH}_\infty^c$ , where  $N$  and  $M$  are in  $\mathcal{MH}_\infty^c$ ,  $M$  is rational, and  $\det(M) \in \mathcal{R}_\infty$  has exactly 1 zeros in  $\overline{\mathbb{C}}_0^+$  and none on the imaginary axis. Define  $U_1$  and  $U_2$  as in the proof of Lemma 4.2.1.4. From (4.7) we obtain

$$\begin{bmatrix} K_0(-\cdot) \\ I_m \end{bmatrix} = \Lambda(-\cdot)^{-1} \begin{bmatrix} U_1(-\cdot) \\ U_2(-\cdot) \end{bmatrix} M(\cdot)^{-1} = V(-\cdot) \begin{bmatrix} U_1(-\cdot) \\ U_2(-\cdot) \end{bmatrix} M(\cdot)^{-1} \quad (4.15)$$

and so

$$\begin{aligned} K_0(-\cdot) &= [V_{11}(-\cdot)U_1(-\cdot) + V_{12}(-\cdot)U_2(-\cdot)] M(\cdot)^{-1} \\ I_m &= [V_{21}(-\cdot)U_1(-\cdot) + V_{22}(-\cdot)U_2(-\cdot)] M(\cdot)^{-1}. \end{aligned}$$

Thus

$$\begin{aligned} K_0(-\cdot) &= (V_{11}(-\cdot)Q(-\cdot) + V_{12}(-\cdot)) U_2(-\cdot)M(\cdot)^{-1} \\ &= (V_{11}(-\cdot)Q(-\cdot) + V_{12}(-\cdot)) (V_{21}(-\cdot)Q(-\cdot) + V_{22}(-\cdot))^{-1}, \end{aligned}$$

where  $Q(-\cdot) := U_1(-\cdot)U_2(-\cdot)^{-1}$ .

Next, we show that  $U_1(-\cdot)$  and  $U_2(-\cdot)$  are right coprime over  $\mathcal{MH}_\infty^c$ .

Since  $M$  and  $N$  are right coprime, there exist  $R$  and  $S$  in  $\mathcal{MH}_\infty^c$  such that  $RM - SN = I$ . Now it is readily verified using  $V(-\cdot)\Lambda(-\cdot) = I$  that

$$V_{11}(-\cdot)U_1(-\cdot) + V_{12}(-\cdot)U_2(-\cdot) = K(-\cdot)M(\cdot) = N(\cdot) \quad (4.16)$$

$$V_{21}(-\cdot)U_1(-\cdot) + V_{22}(-\cdot)U_2(-\cdot) = M(\cdot). \quad (4.17)$$

Consider now

$$\begin{aligned} &[-S(\cdot)V_{12}(-\cdot) + R(\cdot)V_{22}(-\cdot)] U_2(-\cdot) - [S(\cdot)V_{11}(-\cdot) - R(\cdot)V_{21}(-\cdot)] U_1(-\cdot) \\ &= R(\cdot)[V_{21}(-\cdot)U_1(-\cdot) + V_{22}(-\cdot)U_2(-\cdot)] - S(\cdot)[V_{11}(-\cdot)U_1(-\cdot) + V_{12}(-\cdot)U_2(-\cdot)] \\ &= R(\cdot)M(\cdot) - S(\cdot)N(\cdot) \\ &= I. \end{aligned}$$

Thus  $U_1(-\cdot)$  and  $U_2(-\cdot)$  are right coprime over  $\mathcal{MH}_\infty^c$ .

Finally, we show that  $Q(-\cdot) \in H_\infty^c(\mathbb{C}^{\mathbb{P} \times \mathbb{m}})$ , by showing that  $s \mapsto U_2(-s)$  has no zeros in  $\overline{\mathbb{C}_0^+}$ . Proceeding as in the proof of Lemma 4.2.1.4, we have that the sum of the number of zeros of  $s \mapsto \det(V_{22}(-\zeta - s))$  in  $\overline{\mathbb{C}_0^+}$  plus the number of zeros of  $s \mapsto \det(U_2(-\zeta - s))$  in  $\overline{\mathbb{C}_0^+}$  equals the number of zeros of  $s \mapsto \det(V_{21}(-\zeta - s)U_1(-\zeta - s) + V_{22}(-\zeta - s)U_2(-\zeta - s))$  ( $= \det(M(\zeta + s)$ , using (4.17)) in  $\overline{\mathbb{C}_0^+}$ , that is,  $1 + \text{den}(Q) = 1$ , where  $\text{den}(Q)$  denotes the number of zeros of  $s \mapsto \det(U_2(-\zeta - s))$  in  $\overline{\mathbb{C}_0^+}$ . Thus  $Q(-\cdot)$  has no poles in  $\overline{\mathbb{C}_\zeta^+}$ . But  $\zeta > 0$  can be chosen arbitrarily small, which implies that  $Q(-\cdot) \in H_\infty^c(\mathbb{C}^{\mathbb{P} \times \mathbb{m}})$ . ■

Consequently, our main result is the following:

**Theorem 4.3.3** *Suppose that S1-S5 hold. Then  $K(-\cdot) \in H_{\infty, [1]}^c(\mathbb{C}^{\mathbb{P} \times \mathbb{m}})$  and  $\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma$  iff  $K(-\cdot) = R_1(-\cdot)R_2(-\cdot)^{-1}$ , where*

$$\begin{bmatrix} R_1(-\cdot) \\ R_2(-\cdot) \end{bmatrix} = \Lambda(-\cdot)^{-1} \begin{bmatrix} Q(-\cdot) \\ I_m \end{bmatrix}$$

for some  $Q(-\cdot) \in H_\infty^c(\mathbb{C}^{\mathbb{P} \times \mathbb{m}})$  satisfying  $\|Q(i\cdot)\|_\infty \leq 1$ .

## 4.4 A comparison with existing results

It is appropriate to compare the results in Theorem 4.1.4 and Theorem 4.3.3 of this chapter with a corollary to the abstract results from Ball and Helton [5] as stated in Curtain and Ran [17]. Note that for an easier comparison we have chosen the version which is closer to our main result. Denote by  $\hat{\mathcal{A}}(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$  the set  $\{\hat{h} \mid h \in L_1([0, \infty), \mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})\}$ .  $\hat{\mathcal{A}}_1(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$  denotes the set of complex  $\mathfrak{p} \times \mathfrak{m}$  matrix-valued functions  $X(\cdot)$  of a complex variable defined in the closed right half-plane with a decomposition  $X = G_* + F$ , where  $G_*$  is the matrix transfer function of a system of MacMillan degree at most equal to 1, with all its poles in the open right half-plane, and  $F \in \hat{\mathcal{A}}(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$ . The following result follows from Ball and Helton [5].

**Theorem 4.4.1** *Let  $G \in \hat{\mathcal{A}}$  and let  $\sigma_k$ 's denote the Hankel singular values of  $G$ . Then*

$$\inf_{K(\cdot) \in \hat{\mathcal{A}}_1(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})} \|G(i\cdot) + K(i\cdot)\|_\infty = \sigma_{1+1}.$$

*If  $\sigma_{1+1} < \sigma < \sigma_1$ , then there exists a  $\Lambda(\cdot) \in \hat{\mathcal{A}}(\mathbb{C}^{(\mathfrak{p}+\mathfrak{m}) \times (\mathfrak{p}+\mathfrak{m})})$  such that  $K(\cdot) \in \hat{\mathcal{A}}_1(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$  satisfies  $\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma$  iff  $K(\cdot) = R_1(\cdot)R_2(\cdot)^{-1}$ , where*

$$\begin{bmatrix} R_1(\cdot) \\ R_2(\cdot) \end{bmatrix} = \Lambda(\cdot)^{-1} \begin{bmatrix} Q(\cdot) \\ I_{\mathfrak{m}} \end{bmatrix}$$

*for some  $Q(\cdot) \in \hat{\mathcal{A}}(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$  satisfying  $\|Q(i\cdot)\|_\infty \leq 1$ . Moreover,  $\Lambda(\cdot)$  is any element in  $\hat{\mathcal{A}}(\mathbb{C}^{(\mathfrak{p}+\mathfrak{m}) \times (\mathfrak{p}+\mathfrak{m})})$  such that  $\Lambda(\cdot)^{-1} \in \hat{\mathcal{A}}(\mathbb{C}^{(\mathfrak{p}+\mathfrak{m}) \times (\mathfrak{p}+\mathfrak{m})})$  and it satisfies (4.4).*

Notice that our result is for a more general class of transfer functions  $G$  in  $\mathcal{C}(\mathbb{R}, \mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$  with limit 0 at  $\pm i\infty$ , although we cannot exclude the possibility that our result could be deduced from the abstract results of Ball and Helton [5]. Further comparison reveals that our Theorem 4.3.3 only gives sufficient conditions, whereas Theorem 4.4.1 gives *necessary and sufficient* conditions. For the class  $G \in \hat{\mathcal{A}}$  the necessity part is fairly easy to show using Theorem 2.3 of Iftime and Zwart [47] (see also the elementary proof of part  $b. \Rightarrow c.$  of Theorem 3.1 of [47]). This elementary proof relies basically on the existence of a spectral factorization, which is well-known for the Wiener class (see Clancey and Gohberg [10]).

The major difference between Theorem 4.4.1 and our main result is that we have used elementary transparent proofs in contrast to the abstract methods in Ball and Helton [5] which subsequently need to be translated to this problem.

# Chapter 5

## State space solutions

In this chapter, we will show that for the Pritchard-Salamon class and the analytic class, there do exist  $J$ -spectral factors for each class satisfying the assumptions in Chapter 4, hence solving the sub-optimal Hankel norm approximation problem for these classes of infinite-dimensional systems. We will give an explicit formula for a  $J$ -spectral factor  $\Lambda$  in terms of the state-space parameters  $(A, B, C)$  of the original system and we will check that it has all the properties S1-5 required in Chapter 4.

### 5.1 $J$ -spectral factorization for the Pritchard-Salamon class

In this section, we show the existence of a solution to the key  $J$ -spectral factorization problem (satisfying S1-S5 in Chapter 4) for the smooth Pritchard-Salamon class of exponentially stable infinite-dimensional systems with finite-dimensional input and output spaces, in terms of the system parameters  $A, B, C$ . We list the assumptions below:

- P1.  $\Sigma(A, B, C)$  is an exponentially stable, smooth Pritchard-Salamon system.
- P2.  $U = \mathbb{C}^m, Y = \mathbb{C}^p$ .
- P3.  $\sigma_{1+1} < \sigma < \sigma_1$ .

In Curtain and Ran [17] the sub-optimal Hankel norm problem was solved for the

Pritchard-Salamon class by finding a solution to the  $J$ -spectral factorization problem considered in S3 with  $G(s) = C(sI - A)^{-1}B$ . However, there the starting point was to quote a result from Ball and Helton [5], which states that the sub-optimal Hankel norm approximation problem is equivalent to solving the  $J$ -spectral factorization problem in S3. Then a solution is constructed from a given realization of  $G$ . However, if one looks for the result quoted from [5], one realizes that this is not an obvious corollary of the very abstract and general theory in [5]. This motivated the self-contained proofs of the sub-optimal Hankel norm approximation problem in Chapter 4. Here we use just the results from Chapter 4 and verify that all the assumptions are satisfied.

The sub-optimal Nehari problem is a special case of the sub-optimal Hankel norm approximation problem; one seeks  $K(\cdot) \in H_\infty(\mathbb{C}^{\mathbb{P} \times \mathbb{m}})$  satisfying  $\|G(\cdot) + K(\cdot)\|_\infty \leq \sigma$  for a given  $\sigma > \|\Gamma\|$ . In the case that  $\sigma > \|\Gamma\|$ , a solution  $\Lambda$  to the  $J$ -spectral factorization problem S3 was constructed in Curtain and Zwart [26] for the Pritchard-Salamon class.

For the sub-optimal Hankel norm problem we have  $\sigma_{1+1} < \sigma < \sigma_1$ , but nonetheless, the  $\Lambda$  factor is precisely the same. The proof of the following lemma is analogous to that of Lemma 2.11 in Curtain and Zwart [26]:

**Lemma 5.1.1** *Suppose that  $\Sigma(A, B, C)$  satisfies P1-P3 and let  $G(s) = C(sI - A)^{-1}B$ . Then*

$$\Sigma \left( A', [ C' \quad L_C B ], -\frac{1}{\sigma^2} \begin{bmatrix} -CL_B \\ \sigma B' \end{bmatrix} N'_\sigma, \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix} \right)$$

*is an exponentially stable, smooth Pritchard-Salamon system and its transfer operator is  $\Lambda(\cdot)$ , where*

$$\Lambda(s) = \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix} + \frac{1}{\sigma^2} \begin{bmatrix} -CL_B \\ \sigma B' \end{bmatrix} N'_\sigma (sI + A')^{-1} [ C' \quad L_C B ], \quad (5.1)$$

*$N_\sigma = (I - \frac{1}{\sigma^2} L_B L_C)^{-1}$  and  $s \in \overline{\mathbb{C}_0^-}$ . Moreover,*

$$\Sigma \left( A', N'_\sigma [ C' \quad \frac{1}{\sigma} L_C B ], \frac{1}{\sigma^2} \begin{bmatrix} -CL_B \\ \sigma B' \end{bmatrix}, \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix} \right)$$

*is an exponentially stable, smooth Pritchard-Salamon system and its transfer operator is  $V(\cdot)$ , where*

$$V(s) = \begin{bmatrix} I_p & 0 \\ 0 & \frac{1}{\sigma} I_m \end{bmatrix} - \frac{1}{\sigma^2} \begin{bmatrix} -CL_B \\ B' \end{bmatrix} (sI + A')^{-1} N'_\sigma [ C' \quad \frac{1}{\sigma} L_C B ], \quad s \in \overline{\mathbb{C}_0^-}.$$

*Furthermore,*

1.  $\Lambda(\cdot) \in \mathcal{MH}_\infty^c$  is invertible in  $\mathcal{MH}_\infty^c$ , and  $\Lambda(\cdot)^{-1}$  is equal to  $V(\cdot)$ .

2.  $\Lambda$  given by (5.1) is a solution of the  $J$ -spectral factorization problem (4.4) in  $S3$ .

$$3. \lim_{\substack{s \rightarrow \infty \\ s \in \mathbb{C}_0^+}} \Lambda(-s) = \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix}.$$

**Remarks:**

1. Note that while it is algebraically straightforward to verify that  $\Lambda(s)$  solves the  $J$ -spectral factorization problem, it is important to justify the well-posedness of each term at all steps. Use is made of the Lyapunov equations from Lemma 2.3.1, but to make sense of all terms, it was necessary to make use of the smoothing properties of  $L_B$  and  $L_C$  from Lemma 2.3.1. Of course, these smoothing properties are only known to hold for *smooth* Pritchard-Salamon systems.
2. If  $\sigma > \|\Gamma\|$ , it was shown in Curtain and Zwart [26] that  $\Lambda_{11}(\cdot)$ 
  - (a) belongs to  $\mathcal{MH}_\infty^c$ ,
  - (b) is invertible in  $\mathcal{MH}_\infty^c$  and
  - (c)  $\Lambda_{11}(\cdot)^{-1}$  is the transfer operator of an exponentially stable smooth Pritchard-Salamon system.

In the case that  $\sigma_{1+1} < \sigma < \sigma_1$ ,  $\Lambda_{11}(\cdot)^{-1}$  still exists, but it is not stable:  $\Lambda_{11}(\cdot)^{-1}$  is the sum of a stable part in  $H_\infty^c(\mathbb{C}^p \times \mathbb{P})$  and an antistable rational part with at most 1 unstable poles.

Finally we give the main result of this section.

**Theorem 5.1.2** *Suppose that  $\Sigma(A, B, C)$  satisfies P1-P3 and let  $G(s) = C(sI - A)^{-1}B$ . Let  $\Lambda$  be given by (5.1). Then we have:*

$K(\cdot) \in H_{\infty,1}^c(\mathbb{C}^p \times \mathbb{m})$  satisfies  $\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma$  iff  $K(\cdot) = R_1(\cdot)R_2(\cdot)^{-1}$ , where

$$\begin{bmatrix} R_1(\cdot) \\ R_2(\cdot) \end{bmatrix} = \Lambda(\cdot)^{-1} \begin{bmatrix} Q(\cdot) \\ I_m \end{bmatrix}$$

for some  $Q(\cdot) \in H_\infty^c(\mathbb{C}^p \times \mathbb{m})$  satisfying  $\|Q(i\cdot)\|_\infty \leq 1$ .

**Proof** The assumptions S1-S5 in Chapter 4 hold. S1 is satisfied since  $G$  is the transfer operator of an exponentially stable smooth Pritchard-Salamon system (see 2.12) and S2 holds. Finally, Lemma 5.1.1 shows that S3-S5. Now the result follows from Theorem 4.3.3. ■

**Example 2.3.4** (continued) All the results in this section apply to the Example 2.3.4 considered earlier in Chapter 2. ◇

## 5.2 $J$ -spectral factorization for the analytic class

In this section, we show the existence of a solution to the key  $J$ -spectral factorization problem satisfying S1-S5 in Chapter 4 for the exponentially stable analytic class of infinite-dimensional systems discussed in Section 2.4 of Chapter 2 with finite-dimensional input and output spaces: throughout this chapter, we consider the class of systems described by the triple  $(-A, B, C)$  satisfying A1, A2, A3 with  $U = \mathbb{C}^m, Y = \mathbb{C}^p$  (see page 26). We list all the assumptions below:

- A1.  $-A$  is the infinitesimal generator of an exponentially stable analytic semigroup  $\{T(t)\}_{t \geq 0}$  on the Hilbert space  $Z$ .
- A2.  $B \in \mathcal{L}(\mathbb{C}^m, Z_\alpha)$ , where  $\alpha$  is a fixed number in  $(-1, 0]$ .
- A3.  $C \in \mathcal{L}(Z, \mathbb{C}^p)$ .
- A4.  $\sigma_{1+1} < \sigma < \sigma_1$ .

We construct a solution  $\Lambda$  to the  $J$ -spectral factorization problem (??) which has the properties S3-S5. The spectral factor  $\Lambda$  is the same one as in Curtain and Ichikawa [14] (in which they consider the Nehari problem for a similar class with some differences; see also Table 2.1), but for the sub-optimal Hankel norm approximation problem we have  $\sigma_{1+1} < \sigma < \sigma_1$ . Extensive use of the results of Section 2.4 is made to show that all the conditions in Chapter 4 are satisfied. The candidate solution is described by the quadruple

$$\left( -A', [ C' \quad L_C B ], \frac{1}{\sigma^2} \begin{bmatrix} CL_B \\ -\sigma B' \end{bmatrix} N'_\sigma, \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix} \right).$$

First we show that this defines a regular linear system (see Section 2.1), and that its transfer function is invertible.

**Lemma 5.2.1** *Let the triple  $(-A, B, C)$  satisfy A1-A4 and let  $G(s) = C(sI + A)^{-1}B$ . Let  $\beta$  be a fixed number satisfying  $-1 < \beta < \alpha$ . Choose any  $\gamma$  satisfying  $-\beta - \frac{1}{2} < \gamma < \frac{1}{2}$ . Then*

$$\Sigma_1 = \left( -A', [ C' \quad L_C B ], \frac{1}{\sigma^2} \begin{bmatrix} CL_B \\ -\sigma B' \end{bmatrix} N'_\sigma, 0 \right), \tag{5.2}$$

and

$$\Sigma_2 = \left( -A', N'_\sigma [ C' \quad \frac{1}{\sigma} L_C B ], \frac{1}{\sigma^2} \begin{bmatrix} CL_B \\ -B' \end{bmatrix}, 0 \right) \tag{5.3}$$

are regular linear systems with state space  $Z'_\gamma$ , input space  $\mathbb{C}^{\mathfrak{p}+\mathfrak{m}}$ , and output space  $\mathbb{C}^{\mathfrak{p}+\mathfrak{m}}$ . Denote the transfer function of  $\Sigma_1$  by

$$G_1(\cdot) = \frac{1}{\sigma^2} \begin{bmatrix} CL_B \\ -\sigma B' \end{bmatrix} N'_\sigma (\cdot I + A')^{-1} \begin{bmatrix} C' & L_C B \end{bmatrix} \in H_\infty^c \left( \mathbb{C}^{(\mathfrak{p}+\mathfrak{m}) \times (\mathfrak{p}+\mathfrak{m})} \right). \quad (5.4)$$

Then we have

$$\lim_{\substack{s \rightarrow \infty \\ s \in \overline{\mathbb{C}_+^+}}} G_1(s) = 0. \quad (5.5)$$

Furthermore, if

$$\Lambda(-s) := \begin{bmatrix} I_{\mathfrak{p}} & 0 \\ 0 & \sigma I_{\mathfrak{m}} \end{bmatrix} + G_1(s) \text{ for all } s \in \overline{\mathbb{C}_0^+}, \quad (5.6)$$

then  $\Lambda(-\cdot)$  is an element in  $H_\infty^c \left( \mathbb{C}^{(\mathfrak{p}+\mathfrak{m}) \times (\mathfrak{p}+\mathfrak{m})} \right)$ , it is invertible over  $H_\infty^c \left( \mathbb{C}^{(\mathfrak{p}+\mathfrak{m}) \times (\mathfrak{p}+\mathfrak{m})} \right)$  and its inverse is given by  $V(-\cdot)$ , where

$$V(-\cdot) = \begin{bmatrix} I_{\mathfrak{p}} & 0 \\ 0 & \frac{1}{\sigma} I_{\mathfrak{m}} \end{bmatrix} - G_2(\cdot),$$

and  $G_2 \in H_\infty^c \left( \mathbb{C}^{(\mathfrak{p}+\mathfrak{m}) \times (\mathfrak{p}+\mathfrak{m})} \right)$  denotes the transfer function of  $\Sigma_2$ :

$$G_2(s) = \frac{1}{\sigma^2} \begin{bmatrix} CL_B \\ -B' \end{bmatrix} (sI + A')^{-1} N'_\sigma \begin{bmatrix} C' & \frac{1}{\sigma} L_C B \end{bmatrix} \quad (5.7)$$

for all  $s \in \overline{\mathbb{C}_0^+}$ .

**Proof** To make sense of the terms  $B'N'_\sigma$  and  $L_B N'_\sigma$ , we first note by appealing to Lemma 2.4.8 that  $N_\sigma \in \mathcal{L}(Z_\beta)$  and  $N'_\sigma \in \mathcal{L}(Z'_\beta)$  are well-defined for any  $\beta$  satisfying  $-1 < \beta < \alpha$ . From Lemma 2.4.1.2, it follows that  $L_B \in \mathcal{L}(Z'_{-\gamma}, Z)$  for  $\gamma < 1 + 2\alpha$ , and so  $L_B N'_\sigma \in \mathcal{L}(Z'_{-\beta}, Z)$  for  $\beta < 1 + 2\alpha$ . Here we remark that if  $\beta$  satisfies  $-1 < \beta < \alpha$ , then in particular  $\beta < 1 + 2\alpha$ . Since  $B \in \mathcal{L}(\mathbb{C}^{\mathfrak{m}}, Z_\beta)$  for  $-1 < \beta < \alpha$ , we have  $B'N'_\sigma \in \mathcal{L}(Z'_{-\beta}, \mathbb{C}^{\mathfrak{m}})$ . Furthermore, if  $-1 < \beta < \alpha$ , then using Lemma 2.4.2.2, we obtain that  $L_C \in \mathcal{L}(Z_\beta, Z)$ , and so  $L_C B \in \mathcal{L}(\mathbb{C}^{\mathfrak{m}}, Z)$ . Consequently, if  $-1 < \beta < \alpha$ , then we have:

$$\begin{bmatrix} C' & L_C B \end{bmatrix} \in \mathcal{L}(\mathbb{C}^{\mathfrak{p}+\mathfrak{m}}, Z), \quad \begin{bmatrix} CL_B \\ -\sigma B' \end{bmatrix} N'_\sigma \in \mathcal{L}(Z'_{-\beta}, \mathbb{C}^{\mathfrak{p}+\mathfrak{m}}).$$

We now fix a  $\gamma$  satisfying  $-\beta - \frac{1}{2} < \gamma < \frac{1}{2}$ . We remark that such a choice is possible, since  $-1 < \beta$ . The freedom in the choice of  $\gamma$  is depicted in Figure 5.1, where we have one of the possible regions depending on the value of  $\alpha$ .

$$\text{Case 1: } -1 < \alpha < -\frac{1}{2}, \quad \text{Case 2: } \alpha = -\frac{1}{2}, \quad \text{Case 3: } -\frac{1}{2} < \alpha \leq 0.$$

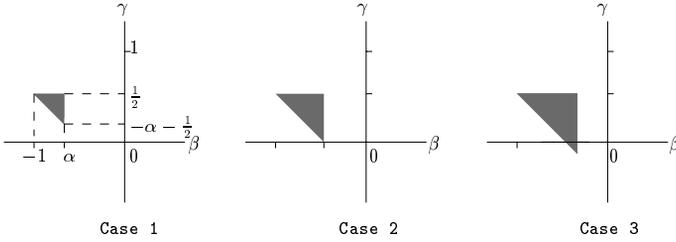


Figure 5.1: The possible cases.

From Theorem 2.4.4 with  $\alpha_B = 0$  and  $\alpha_C = -\beta$  (which satisfies the condition  $\alpha_B \leq \alpha_C < \alpha_B + 1$ ), it follows that  $\Sigma_1$  is a regular linear system on  $Z'_\gamma$  with a transfer function  $G_1(\cdot)$  given by (5.4), which satisfies  $\lim_{\substack{s \rightarrow \infty \\ s \in \mathbb{C}_+}} G_1(s) = 0$ .

Similarly, it can be seen, again using Theorem 2.4.4, that  $\Sigma_2$  is a regular linear system on  $Z'_\gamma$  with a transfer function  $G_2(\cdot)$  given by (5.7).

Finally we have to show that  $\Lambda(\cdot)V(\cdot) = I_{p+m} = V(\cdot)\Lambda(\cdot)$ . Consider

$$\begin{aligned} & \Lambda(-s)V(-s) - \begin{bmatrix} I_p & 0 \\ 0 & I_m \end{bmatrix} \\ &= \frac{1}{\sigma^2} \begin{bmatrix} CL_B \\ -\sigma B' \end{bmatrix} N'_\sigma (sI + A')^{-1} \begin{bmatrix} C' & \frac{1}{\sigma} L_C B \end{bmatrix} \\ & \quad - \frac{1}{\sigma^2} \begin{bmatrix} CL_B \\ -\sigma B' \end{bmatrix} (sI + A')^{-1} N'_\sigma \begin{bmatrix} C' & \frac{1}{\sigma} L_C B \end{bmatrix} \\ & \quad - \frac{1}{\sigma^4} \begin{bmatrix} CL_B \\ -\sigma B' \end{bmatrix} N'_\sigma (sI + A')^{-1} \begin{bmatrix} C' & \frac{1}{\sigma} L_C B \end{bmatrix} \\ & \quad \begin{bmatrix} CL_B \\ -\sigma B' \end{bmatrix} (sI - A')^{-1} N'_\sigma \begin{bmatrix} C' & \frac{1}{\sigma} L_C B \end{bmatrix}. \end{aligned}$$

We have

$$\begin{bmatrix} C' & \frac{1}{\sigma} L_C B \end{bmatrix} \begin{bmatrix} CL_B \\ -\sigma B' \end{bmatrix} = C'CL_B - L_CBB'.$$

Furthermore, it can be verified that for  $z \in Z'_{-\beta}$ , where  $-1 < \beta < \alpha$ , there holds

$$\begin{aligned} C'CL_Bz - L_CBB'z &= A' L_C L_B z - L_C L_B A' z \\ &= \sigma^2 (N'_\sigma)^{-1} (sI + A') z - \sigma^2 (sI + A') (N'_\sigma)^{-1} z \end{aligned}$$

in  $Z'_{-\beta-1}$ . This can be checked by using the Lyapunov equations (2.17) and

(2.25), and Lemmas 2.4.1.2 and 2.4.2.2. Consequently, we obtain

$$\begin{aligned}
 \Lambda(-s)V(-s) - \begin{bmatrix} I_p & 0 \\ 0 & I_m \end{bmatrix} &= \frac{1}{\sigma^2} \begin{bmatrix} CL_B \\ -\sigma B' \end{bmatrix} N'_\sigma (sI + A')^{-1} \begin{bmatrix} C' & \frac{1}{\sigma} L_C B \end{bmatrix} \\
 &\quad - \frac{1}{\sigma^2} \begin{bmatrix} CL_B \\ -\sigma B' \end{bmatrix} (sI + A')^{-1} N'_\sigma \begin{bmatrix} C' & \frac{1}{\sigma} L_C B \end{bmatrix} \\
 &\quad - \frac{1}{\sigma^4} \begin{bmatrix} CL_B \\ -\sigma B' \end{bmatrix} N'_\sigma (sI + A')^{-1} \\
 &\quad \left[ \sigma^2 (N'_\sigma)^{-1} (sI + A') - \sigma^2 (sI + A') (N'_\sigma)^{-1} \right] \\
 &\quad (sI - A')^{-1} N'_\sigma \begin{bmatrix} C' & \frac{1}{\sigma} L_C B \end{bmatrix} \\
 &= 0.
 \end{aligned}$$

■

**Remark:** In several places in this section we need to verify various identities which, taken algebraically, are easy to verify. However, when dealing with unbounded operators, it is necessary to check at every step that all the terms are well-defined and in the correct spaces. We do this by making extensive use of Lemmas 2.4.1, 2.4.2 and Corollary 2.4.5. We have made this explicit in the first part of this proof of Lemma 5.2.1.

**Lemma 5.2.2** *Suppose that the triple  $(-A, B, C)$  satisfies A1-A4 and let  $G(s) = C(sI + A)^{-1}B$ . If  $\Lambda$  is given by (5.6), then  $\Lambda$  is a solution to the  $J$ -spectral factorization problem (4.4) in S3.*

**Proof** This is proved by a tedious but straightforward substitution, again verifying that all the terms are well-defined at each step. For a step by step calculation bounded  $B$  and  $C$ , see Curtain and Zwart [27]. ■

**Remark:** In the case that  $\sigma_{1+1} < \sigma < \sigma_1$ ,  $\Lambda_{11}(\cdot)^{-1}$  exists, but it is not stable:  $\Lambda_{11}(\cdot)^{-1}$  can be written as the sum of a stable part in  $H_\infty(\mathbb{C}^{\mathbb{P}^p})$  and an antistable rational part with at most 1 unstable poles.

Finally we give the main result of this section.

**Theorem 5.2.3** *Suppose that  $(-A, B, C)$  is a triple satisfying the assumptions A1-A4 listed at the beginning of this section and let  $G(s) = C(sI + A)^{-1}B$ . Let  $\Lambda$  be given by (5.6).*

$K(\cdot) \in H_{\infty,1}^c(\mathbb{C}^{\mathbb{P}^m})$  satisfies  $\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma$  iff  $K(\cdot) = R_1(\cdot)R_2(\cdot)^{-1}$ , where

$$\begin{bmatrix} R_1(\cdot) \\ R_2(\cdot) \end{bmatrix} = \Lambda(\cdot)^{-1} \begin{bmatrix} Q(\cdot) \\ I_m \end{bmatrix}$$

for some  $Q(\cdot) \in H_\infty^c(\mathbb{C}^{\mathbb{P} \times \mathbb{m}})$  satisfying  $\|Q(i\cdot)\|_\infty \leq 1$ .

**Proof** The assumptions S1-S5 in Chapter 4 hold. S1 is satisfied (see Theorem 2.4.4) and S2 holds. That S3, S4 and S5 hold follows from Lemma 5.2.1 and Lemma 5.2.2. The result now follows from Theorem 4.3.3. ■

**Example 2.4.10** (continued) We remark that all the results in this section apply to the Example 2.4.10 considered earlier in Chapter 2. ◇

## Chapter 6

# The non-exponentially stable case

In the previous chapters the sub-optimal Hankel norm approximation problem was solved for various classes of infinite-dimensional systems under the assumption that  $A$  generates an *exponentially stable* strongly continuous semigroup. However, there exists an important class of systems with a transfer function  $G \in H_\infty(\mathbb{C}^{\times m})$ , for which  $A$  does not generate an exponentially stable semigroup (see Examples 3.1.1, 3.1.2 and 3.2.4). In [37], approximating solutions to the optimal Hankel norm approximation problem were obtained without assuming exponential stability, but only for the case that the Hankel operator is nuclear, which is a strong assumption. In this chapter, we will solve the sub-optimal Hankel norm approximation problem for non-exponentially stable infinite-dimensional systems in terms of a solution to the sub-optimal Hankel norm approximation problem for an exponentially stable system which is obtained by shifting the generator of the semigroup of the original system.

The outline is as follows. In section 6.1 we will prove a useful lemma which will be used later and in section 6.2 we will introduce our class of systems in terms of assumptions on  $h$  and its Fourier transform, that is, without the use of a state-space realization. We will prove a crucial convergence result for this class and then prove our main Theorem 6.2.4. Finally in the last section 6.3, we will apply the results from the previous section and the results from Chapter 5 to obtain state-space solutions to the sub-optimal Hankel norm approximation problem for non-exponentially stable infinite-dimensional state-linear systems and give an illustrative example.

## 6.1 A few useful results

First we will prove a few elementary results which will be used later in this chapter. The key to the proof of the new result (Theorem 6.2.4) is Corollary 6.1.2 which is an easy consequence of the following lemma.

**Lemma 6.1.1** *If  $G \in H_\infty(\mathbb{C}^{\times m})$  and  $\omega \mapsto G(i\omega) : \mathbb{R} \rightarrow \mathbb{C}^{\times m}$  is uniformly continuous, then given any  $\epsilon > 0$ , there exists a  $\delta > 0$  such that*

$$\sup_{\omega \in \mathbb{R}} \|G(i\omega) - G(\zeta + i\omega)\| < \epsilon \text{ for all } \zeta \text{ satisfying } 0 \leq \zeta \leq \delta.$$

**Proof** It follows from Theorem 5.18 (page 96, M. Rosenblum and J. Rovnyak [69]) that

$$G(\zeta + i\omega) = \frac{\zeta}{\pi} \int_{-\infty}^{\infty} \frac{G(it)}{(t - \omega)^2 + \zeta^2} dt, \quad \zeta > 0.$$

Since for  $\zeta > 0$ ,

$$\frac{\zeta}{\pi} \int_{-\infty}^{\infty} \frac{1}{(t - \omega)^2 + \zeta^2} dt = 1,$$

and we have

$$\begin{aligned} \|G(i\omega) - G(\zeta + i\omega)\| &= \left\| \frac{\zeta}{\pi} \int_{-\infty}^{\infty} \frac{G(it)}{(t - \omega)^2 + \zeta^2} dt - G(i\omega) \right\| \\ &= \left\| \frac{\zeta}{\pi} \int_{-\infty}^{\infty} \frac{G(it) - G(i\omega)}{(t - \omega)^2 + \zeta^2} dt \right\|. \end{aligned}$$

Choose a  $\eta > 0$  such that

$$\|G(it) - G(i\omega)\| < \frac{\epsilon}{2} \text{ for all } t \text{ and } \omega \text{ satisfying } |t - \omega| < \eta.$$

Now choose a  $\delta > 0$  such that for any  $\zeta$  satisfying  $0 \leq \zeta \leq \delta$ , we have

$$\left\| \frac{\zeta}{\pi} \int_{\mathbb{R} \setminus [\omega - \eta, \omega + \eta]} \frac{1}{(t - \omega)^2 + \zeta^2} dt \right\| < \frac{\epsilon}{4 \|G(i \cdot)\|_\infty}.$$

Thus

$$\begin{aligned} & \|G(i\omega) - G(\zeta + i\omega)\| \\ &= \left\| \frac{\zeta}{\pi} \int_{-\infty}^{\infty} \frac{G(it) - G(i\omega)}{(t - \omega)^2 + \zeta^2} dt \right\| \\ &\leq \frac{\zeta}{\pi} \int_{\omega - \eta}^{\omega + \eta} \frac{\|G(it) - G(i\omega)\|}{(t - \omega)^2 + \zeta^2} dt + \frac{\zeta}{\pi} \int_{\mathbb{R} \setminus [\omega - \eta, \omega + \eta]} \frac{\|G(it) - G(i\omega)\|}{(t - \omega)^2 + \zeta^2} dt \\ &\leq \frac{\zeta}{\pi} \int_{\omega - \eta}^{\omega + \eta} \frac{\frac{\epsilon}{2}}{(t - \omega)^2 + \zeta^2} dt + \frac{\zeta}{\pi} \int_{\mathbb{R} \setminus [\omega - \eta, \omega + \eta]} \frac{2 \|G(i \cdot)\|_\infty}{(t - \omega)^2 + \zeta^2} dt \\ &\leq \frac{\epsilon}{2} 1 + 2 \|G(i \cdot)\|_\infty \frac{\epsilon}{4 \|G(i \cdot)\|_\infty} = \epsilon. \end{aligned}$$

Since the choice of  $\omega$  is arbitrary, this completes the proof.  $\blacksquare$

**Corollary 6.1.2** *If  $G \in H_\infty(\mathbb{C}^{\mathbb{P}^{\times m}})$  and  $G(i\cdot) : \mathbb{R} \rightarrow \mathbb{C}^{\mathbb{P}^{\times m}}$  is continuous and has limits  $G_{\pm\infty}$  at  $\pm\infty$ , then given any  $\epsilon > 0$ , there exists a  $\delta > 0$  such that  $\sup_{\omega \in \mathbb{R}} \|G(i\omega) - G(\zeta + i\omega)\| < \epsilon$  whenever  $0 \leq \zeta \leq \delta$ .*

**Proof** Given any  $\epsilon > 0$ , there exist  $M_1 > 0$  and  $M_2 > 0$  such that

$$\sup_{\omega \in [M_2, \infty)} \|G(i\omega) - G_\infty\| < \frac{\epsilon}{2}, \quad \text{and} \quad (6.1)$$

$$\sup_{\omega \in (-\infty, M_1]} \|G(i\omega) - G_{-\infty}\| < \frac{\epsilon}{2}. \quad (6.2)$$

Moreover, since  $G(i\cdot) : \mathbb{R} \rightarrow \mathbb{C}^{\mathbb{P}^{\times m}}$  is continuous, it is uniformly continuous in  $[M_1 - 1, M_2 + 1]$ , and so given any  $\epsilon > 0$ , there exists a  $\delta$  such that  $1 > \delta > 0$  and whenever  $\omega_1$  and  $\omega_2$  lie in  $[M_1 - 1, M_2 + 1]$  and  $|\omega_1 - \omega_2| < \delta$ ,

$$\|G(i\omega_1) - G(i\omega_2)\| < \epsilon. \quad (6.3)$$

Thus it follows from (6.1), (6.2) and (6.3) that whenever  $\omega_1$  and  $\omega_2$  belong to  $\mathbb{R}$  and  $|\omega_1 - \omega_2| < \delta$ , then  $\|G(i\omega_1) - G(i\omega_2)\| < \epsilon$ . Hence  $G(i\cdot) : \mathbb{R} \rightarrow \mathbb{C}^{\mathbb{P}^{\times m}}$  is uniformly continuous, and so the result follows from Lemma 6.1.1.  $\blacksquare$

## 6.2 The class of systems and the main result

The specific class of systems we consider in this chapter is defined below:

- B1. 1.  $h \in L_2([0, \infty), \mathbb{C}^{\mathbb{P}^{\times m}})$ .  
 2.  $G(i\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{C}^{\mathbb{P}^{\times m}})$  and has a unique limit  $G_\infty$  at  $\pm\infty$ , where  $G := \mathcal{F}(h)$ , where  $\mathcal{F}$  denotes the Fourier(-Plancherel) transform.
- B2. There exists a  $\delta > 0$ , such that for every  $\zeta$  satisfying  $0 < \zeta \leq \delta$ ,
1.  $G(\zeta + \cdot) \in H_\infty(\mathbb{C}^{\mathbb{P}^{\times m}})$  has a compact Hankel operator  $\Gamma^{[\zeta]}$ , with singular values  $\sigma_1^{[\zeta]} \geq \sigma_2^{[\zeta]} \geq \dots$ , and
  2. if  $\sigma_{1+1}^{[\zeta]} < \sigma < \sigma_1^{[\zeta]}$ , then  $G(\zeta + \cdot)$  has a solution to the sub-optimal Hankel norm approximation problem:  $K^{[\zeta]}(-\cdot) \in H_{\infty,1}(\mathbb{C}^{\mathbb{P}^{\times m}})$  such that  $\|G(\zeta + i\cdot) + K^{[\zeta]}(i\cdot)\|_\infty \leq \sigma$ .

We remark that 2 of B1 implies that  $G(i \cdot) \in L_\infty(\mathbb{R}, \mathbb{C}^{\mathbb{P} \times \mathbb{m}})$ . Consequently, from Proposition 8 (page 224, Keulen [83]) it follows that  $\Gamma$  is a bounded operator,  $G \in H_\infty(\mathbb{C}^{\mathbb{P} \times \mathbb{m}})$  and  $(H_G[\mathcal{F}(u)])(i\omega) = [\mathcal{F}(\Gamma u)](i\omega)$  for almost all  $\omega \in \mathbb{R}$ , where  $H_G$  is the frequency domain Hankel operator with symbol  $G$ . Furthermore, we note that from 2, by applying Theorem 3.0.11 (in Chapter 3) we obtain that the Hankel operator with symbol  $G$  is compact.

We will now prove a few properties of the class of systems that we consider which will be used in the proving our main Theorem 6.2.4. If  $0 < \zeta \leq \delta$ , then the time-domain Hankel operator corresponding to  $G(\zeta + \cdot) \in H_\infty^c(\mathbb{C}^{\mathbb{P} \times \mathbb{m}})$ , that is, with kernel  $h^{[\zeta]}(\cdot) = e^{-\zeta} h(\cdot) \in L_1([0, \infty), \mathbb{C}^{\mathbb{P} \times \mathbb{m}})$ , is denoted by  $\Gamma^{[\zeta]}$ , and the  $1^{th}$  Hankel singular value is denoted by  $\sigma_1^{[\zeta]}$ . We note that since  $h \in L_2([0, \infty), \mathbb{C}^{\mathbb{P} \times \mathbb{m}})$  and  $e^{-\zeta} \in L_2(0, \infty)$ , it follows from the Cauchy-Schwarz inequality that  $h^{[\zeta]}$  is integrable.

**Lemma 6.2.1** *If an infinite-dimensional system satisfies the assumptions B1 and B2, then*

1.  $[\Gamma^{[\zeta]}]^* \Gamma^{[\zeta]} \rightarrow \Gamma^* \Gamma$  uniformly as  $\zeta \rightarrow 0$ ,
2. Given any  $1 \in \mathbb{N}$ ,  $\sigma_1^{[\zeta]} \rightarrow \sigma_1$  as  $\zeta \rightarrow 0$ .

**Proof** 1. Let  $S_1^{[\zeta]} : L_2([0, \infty), \mathbb{C}^{\mathbb{m}}) \rightarrow L_2([0, \infty), \mathbb{C}^{\mathbb{m}})$  be the multiplication operator by  $e^{-\zeta t}$ : If  $u \in L_2([0, \infty), \mathbb{C}^{\mathbb{m}})$ , then

$$[S_1^{[\zeta]} u](t) = e^{-\zeta t} u(t) \text{ for all } t \geq 0.$$

Similarly, let  $S_2^{[\zeta]} : L_2([0, \infty), \mathbb{C}^{\mathbb{P}}) \rightarrow L_2([0, \infty), \mathbb{C}^{\mathbb{P}})$  be the multiplication operator by  $e^{-\zeta t}$ . Then it can be checked that  $\Gamma^{[\zeta]} = S_2^{[\zeta]} \Gamma S_1^{[\zeta]}$ .

**Sublemma 6.2.2**  $S_1^{[\zeta]} = [S_1^{[\zeta]}]^* \rightarrow I_m$  strongly as  $\zeta \rightarrow 0$ , and  $S_2^{[\zeta]} = [S_2^{[\zeta]}]^* \rightarrow I_p$  strongly as  $\zeta \rightarrow 0$ .

**Proof** Let  $0 \neq u \in L_2([0, \infty), \mathbb{C}^{\mathbb{m}})$ . Given any  $\epsilon > 0$ , choose a  $M > 0$  such that

$$\int_M^\infty \|u(t)\|^2 dt < \frac{\epsilon^2}{2}.$$

Now choose a  $\delta > 0$  such that  $0 \leq \zeta < \delta$  implies that

$$\sup_{t \in [0, M]} |e^{-\zeta t} - 1| < \frac{\epsilon}{\sqrt{2} \|u\|_2}.$$

Thus

$$\begin{aligned}
& \left\| S_1^{[\zeta]} u - u \right\|^2 \\
&= \int_0^\infty \left\| [e^{-\zeta t} - 1] u(t) \right\|^2 dt \\
&\leq \left[ \sup_{t \in [0, M]} |e^{-\zeta t} - 1| \right]^2 \int_0^M \|u(t)\|^2 dt + \int_M^\infty |e^{-\zeta t} - 1|^2 \|u(t)\|^2 dt \\
&\leq \frac{\epsilon^2}{2} + \int_M^\infty \|u(t)\|^2 dt \leq \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2} = \epsilon^2.
\end{aligned}$$

Hence  $S_1^{[\zeta]} = [S_1^{[\zeta]}]^* \rightarrow I_m$  strongly as  $\zeta \rightarrow 0$ . Similarly,  $S_2^{[\zeta]} = [S_2^{[\zeta]}]^* \rightarrow I_p$  strongly as  $\zeta \rightarrow 0$  strongly. ■

Since  $[\Gamma^{[\zeta]}]^* \Gamma^{[\zeta]} = S_1^{[\zeta]} \Gamma^* S_2^{[\zeta]} S_2^{[\zeta]} \Gamma S_1^{[\zeta]}$ , we have

$$[\Gamma^{[\zeta]}]^* \Gamma^{[\zeta]} = S_1^{[\zeta]} \left[ \Gamma^* S_2^{[\zeta]} S_2^{[\zeta]} \Gamma - \Gamma^* \Gamma \right] S_1^{[\zeta]} + S_1^{[\zeta]} \Gamma^* \Gamma S_1^{[\zeta]}. \quad (6.4)$$

Defining  $K = [\Gamma^* \Gamma]^{\frac{1}{2}}$ , we have that  $K$  is compact, since  $K^2$  is compact and  $K$  is self-adjoint (using Exercise 18, page 127, Gohberg and Goldberg [40]). We will use the following (Exercise 6.6', page 136, Weidmann [85]).

**Sublemma 6.2.3** *Let  $\mathcal{U}$ ,  $\mathcal{V}$ ,  $\mathcal{W}$  be Hilbert spaces. Suppose that  $T_n, T \in \mathcal{L}(\mathcal{V}, \mathcal{W})$  and that  $T_n \rightarrow T$  strongly. If  $S \in \mathcal{L}(\mathcal{U}, \mathcal{V})$  is compact, then  $T_n S \rightarrow TS$  uniformly.*

$S_1^{[\zeta]} K \rightarrow K$  uniformly, using the Sublemmas 6.2.2 and 6.2.3. But  $K S_1^{[\zeta]} = [S_1^{[\zeta]} K]^*$ , and  $[S_1^{[\zeta]} K]^* \rightarrow K^* (= K)$  uniformly, and so  $K S_1^{[\zeta]} \rightarrow K$  uniformly. Consequently,

$$S_1^{[\zeta]} \Gamma^* \Gamma S_1^{[\zeta]} \rightarrow \Gamma^* \Gamma \text{ uniformly as } \zeta \rightarrow 0. \quad (6.5)$$

Using the Sublemmas 6.2.2 and 6.2.3, we obtain that  $S_2^{[\zeta]} \Gamma \rightarrow \Gamma$  uniformly. Thus  $\Gamma^* S_2^{[\zeta]} = [S_2^{[\zeta]} \Gamma]^* \rightarrow \Gamma^*$  uniformly. As a result,  $\Gamma^* S_2^{[\zeta]} S_2^{[\zeta]} \Gamma \rightarrow \Gamma^* \Gamma$  uniformly, and since  $\left\| S_1^{[\zeta]} \right\| \leq 1$ , we have

$$S_1^{[\zeta]} \left[ \Gamma^* S_2^{[\zeta]} S_2^{[\zeta]} \Gamma - \Gamma^* \Gamma \right] S_1^{[\zeta]} \rightarrow 0 \text{ uniformly as } \zeta \rightarrow 0. \quad (6.6)$$

From (6.4), (6.6) and (6.5), it follows that  $[\Gamma^{[\zeta]}]^* \Gamma^{[\zeta]} \rightarrow \Gamma^* \Gamma$  uniformly as  $\zeta \rightarrow 0$ .

2. Since  $[\Gamma^{[\zeta]}]^* \Gamma^{[\zeta]} \rightarrow \Gamma^* \Gamma$  uniformly as  $\zeta \rightarrow 0$ ,  $\sigma_1^{[\zeta]} \rightarrow \sigma_1$  as  $\zeta \rightarrow 0$  (using Corollary 4.(a), page 1090, Dunford and Schwartz [30]). ■

Finally, we will prove our main result about the existence of solutions to the sub-optimal Hankel norm approximation problem.

**Theorem 6.2.4** *If an infinite-dimensional system satisfies the assumptions B1 and B2, and if  $\sigma_{1+1} < \sigma < \sigma_1$ , then there exists a  $\zeta_0 > 0$  such that for every  $\zeta$  satisfying  $0 < \zeta < \zeta_0$ ,  $\|G(i\cdot) + K^{[\zeta]}(i\cdot)\|_\infty \leq \sigma$ .*

**Proof** 1. Let  $\epsilon = \frac{\sigma - \sigma_{1+1}}{2} > 0$ . Choose a  $\delta_1 > 0$  small enough so that whenever  $0 \leq \zeta < \delta_1$ ,

$$\sup_{\omega \in \mathbb{R}} \|G(i\omega) - G(\zeta + i\omega)\| < \epsilon.$$

This can be done, owing to assumption B1 and Corollary 6.1.2.

2. Next choose a  $\delta_2 > 0$ , such that whenever  $0 \leq \zeta < \delta_2$ , we have (see Lemma 6.2.1.2)

$$\sigma_{1+1}^{[\zeta]} < \frac{\sigma_{1+1} + \sigma}{2} < \sigma_1^{[\zeta]}.$$

3. Let  $\zeta_0 = \min\{\delta_1, \delta_2\}$  and consider any  $\zeta$  satisfying  $0 < \zeta < \zeta_0$ . From B3, we have the existence of a  $K^{[\zeta]}(\cdot) \in H_{\infty, [1]}^c(\mathbb{C}^{p \times m})$  such that

$$\sup_{\omega \in \mathbb{R}} \|G(i\omega + \zeta) + K^{[\zeta]}(i\omega)\| \leq \frac{\sigma + \sigma_{1+1}}{2}.$$

Thus,

$$\begin{aligned} \|G(i\omega) + K^{[\zeta]}(i\omega)\| &= \|G(i\omega) - G(i\omega + \zeta) + G(i\omega + \zeta) + K^{[\zeta]}(i\omega)\| \\ &\leq \|G(i\omega) - G(i\omega + \zeta)\| + \|G(i\omega + \zeta) + K^{[\zeta]}(i\omega)\| \\ &\leq \frac{\sigma - \sigma_{1+1}}{2} + \frac{\sigma_{1+1} + \sigma}{2} \\ &= \sigma. \end{aligned}$$

This completes the proof. ■

### 6.3 State-space solutions

We now apply the results from the previous section and from Sections 5.1 and 5.2 of Chapter 5 to obtain explicit formulae for the solutions of the sub-optimal Hankel norm approximation problem in the case when  $A$  is not necessarily exponentially stable. We consider the following two classes:

**Class A:** The triple  $(A, B, C)$  satisfies the following assumptions:

1.  $\Sigma(A, B, C)$  is a smooth Pritchard-Salamon system with input space  $\mathbb{C}^m$  and output space  $\mathbb{C}^p$ ,
2. The impulse response  $h$  satisfies B1,
3. For every  $\zeta > 0$ ,  $A - \zeta I$  is the infinitesimal generator of an exponentially stable, strongly continuous semigroup.

**Remarks:**

1. We note that assumption 3 is satisfied if either of the following is true:

ℳ1.  $\sigma(A) \cap \mathbb{C}_0^+$  is empty and  $A$  satisfies the spectrum determined growth assumption<sup>1</sup>.

ℳ2.  $A$  generates a contraction semigroup<sup>2</sup>.

We note that from Exercise 2.4 (page 81, Curtain and Zwart [27]), it follows that if  $A$  generates the semigroup  $\{T(t)\}_{t \geq 0}$ , then  $A - \zeta I$  is the infinitesimal generator of the strongly continuous semigroup  $\{e^{-\zeta t} T(t)\}_{t \geq 0}$  with a growth bound equal to the sum of the growth bound of  $\{T(t)\}_{t \geq 0}$  and  $-\zeta$ . Thus if either ℳ1 or ℳ2 is satisfied, then for every  $\zeta > 0$ ,  $A - \zeta I$  generates the exponentially stable semigroup  $\{e^{-\zeta t} T(t)\}_{t \geq 0}$ .

2. If the triple  $(A, B, C)$  belongs to Class A, then the system  $\Sigma(A - \zeta I, B, C)$  is an exponentially stable smooth Pritchard-Salamon system. We denote the controllability map of this system by  $\mathcal{B}^{[\zeta]}$ , the observability map by  $\mathcal{C}^{[\zeta]}$ , the Hankel operator by  $\Gamma^{[\zeta]}$ , and the 1<sup>th</sup> Hankel singular value by  $\sigma^{[\zeta]}$ . If  $\sigma_{1+1}^{[\zeta]} < \sigma < \sigma_1^{[\zeta]}$ , let  $N_\sigma^{[\zeta]} := \left[ I - \sigma^{-2} L_B^{[\zeta]} L_C^{[\zeta]} \right]^{-1}$ , where  $L_B^{[\zeta]} := \mathcal{B}^{[\zeta]} [\mathcal{B}^{[\zeta]}]'$  and  $L_C^{[\zeta]} := [\mathcal{C}^{[\zeta]}]'$ .

<sup>1</sup> $A$  is said to satisfy the *spectrum determined growth assumption* if the growth bound of the semigroup,  $\omega_0$  equals  $\sup_{\lambda \in \sigma(A)} \operatorname{Re}(\lambda)$

<sup>2</sup>That is,  $\|T(t)\| \leq 1$  for all  $t \geq 0$

**Class B:** The triple  $(A, B, C)$  satisfies the following assumptions:

1.  $-A$  is the infinitesimal generator of an analytic semigroup  $\{T(t)\}_{t \geq 0}$  on the Hilbert space  $Z$ ,  $B \in \mathcal{L}(\mathbb{C}^m, Z_\alpha)$ , where  $-1 < \alpha \leq 0$ , and  $C \in \mathcal{L}(Z, \mathbb{C}^p)$ ,
2. The well-posed linear system<sup>3</sup> given by  $(-A, B, C)$  has an impulse response  $h$  that satisfies B1,
3. For every  $\zeta > 0$ ,  $-A - \zeta I$  is the infinitesimal generator of an exponentially stable, strongly continuous semigroup.

**Remarks:**

1. We note that assumption 3 is satisfied if either of the following is true:
  - ℑ1.  $\sigma(-A) \cap \mathbb{C}_0^+$  is empty and  $-A$  satisfies the spectrum determined growth assumption.
  - ℑ2.  $-A$  generates a contraction semigroup.

We note that from Exercise 2.4 (page 81, Curtain and Zwart [27]), it follows that  $-A - \zeta I$  is the infinitesimal generator of the strongly continuous semigroup  $\{e^{-\zeta t} T(t)\}_{t \geq 0}$  with a growth bound equal to the sum of the growth bound of  $\{T(t)\}_{t \geq 0}$  and  $-\zeta$ . Thus if either ℑ1 or ℑ2 is satisfied, then for every  $\zeta > 0$ ,  $-A - \zeta I$  generates the exponentially stable semigroup  $\{e^{-\zeta t} T(t)\}_{t \geq 0}$ . Furthermore, from Corollary 2.2 (page 81, Pazy [63]) we obtain that in fact  $-A - \zeta I$  generates an *analytic* semigroup.

2. If the triple  $(A, B, C)$  belongs to Class B, then it is easy to see that the system given by the triple  $(-A - \zeta I, B, C)$  is a regular well-posed linear system satisfying the assumptions A1, A2, A3 listed in Section 2.4 of Chapter 2. We denote the controllability map of this system by  $\mathcal{B}^{[\zeta]}$ , the observability map by  $\mathcal{C}^{[\zeta]}$ , the Hankel operator by  $\Gamma^{[\zeta]}$ , and the 1<sup>th</sup> Hankel singular value by  $\sigma^{[\zeta]}$ . If  $\sigma_{1+1}^{[\zeta]} < \sigma < \sigma_1^{[\zeta]}$ , let  $N_\sigma^{[\zeta]} := [I - \sigma^{-2} L_B^{[\zeta]} L_C^{[\zeta]}]^{-1}$ , where  $L_B^{[\zeta]} := \mathcal{B}^{[\zeta]} [\mathcal{B}^{[\zeta]}]'$  and  $L_C^{[\zeta]} := [\mathcal{C}^{[\zeta]}]'$   $\mathcal{C}^{[\zeta]}$ .

**Theorem 6.3.1** *Suppose that the triple  $(A, B, C)$  is in Class A and  $G(s) = C(sI - A)^{-1}B$ . Let  $\sigma_{1+1} < \sigma < \sigma_1$ . If  $Q(\cdot) \in H_\infty(\mathbb{C}^p \times \mathbb{C}^m)$ , and  $\|Q(i \cdot)\|_\infty \leq 1$ , then there exists a  $\delta > 0$  such that for every  $\zeta$  satisfying  $0 < \zeta < \delta$ ,*

$$K^{[\zeta]}(\cdot) = R_1^{[\zeta]}(\cdot) [R_2^{[\zeta]}(\cdot)]^{-1},$$

where

$$\begin{bmatrix} R_1^{[\zeta]}(\cdot) \\ R_2^{[\zeta]}(\cdot) \end{bmatrix} = [\Lambda^{[\zeta]}(\cdot)]^{-1} \begin{bmatrix} Q(\cdot - \zeta) \\ I_m \end{bmatrix}$$

and

$$\Lambda^{[\zeta]}(\cdot) = \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix} + \sigma^{-2} \begin{bmatrix} -CL_B^{[\zeta]} \\ \sigma B' \end{bmatrix} \left[ N_\sigma^{[\zeta]} \right]' (I + A' - \zeta I)^{-1} \begin{bmatrix} C' & L_C^{[\zeta]} B \end{bmatrix}$$

is such that  $K^{[\zeta]}(\cdot) \in H_{\infty, [1]}^c(\mathbb{C}^{\mathbb{P} \times \mathbb{m}})$  and  $\|G(i \cdot) + K^{[\zeta]}(i \cdot)\|_\infty \leq \sigma$ .

**Proof** This is a consequence of Theorem 5.1.2 and Theorem 6.2.4.  $\blacksquare$

**Theorem 6.3.2** *Suppose that the triple  $(A, B, C)$  is in Class B and  $G(s) = C(sI + A)^{-1}B$ . Let  $\sigma_{1+1} < \sigma < \sigma_1$ . If  $Q(\cdot) \in H_\infty(\mathbb{C}^{\mathbb{P} \times \mathbb{m}})$ , and  $\|Q(i \cdot)\|_\infty \leq 1$ , then there exists a  $\delta > 0$  such that for every  $\zeta$  satisfying  $0 < \zeta < \delta$ ,*

$$K^{[\zeta]}(\cdot) = R_1^{[\zeta]}(\cdot) \left[ R_2^{[\zeta]}(\cdot) \right]^{-1},$$

where

$$\begin{bmatrix} R_1^{[\zeta]}(\cdot) \\ R_2^{[\zeta]}(\cdot) \end{bmatrix} = \left[ \Lambda^{[\zeta]}(\cdot) \right]^{-1} \begin{bmatrix} Q(\cdot - \zeta) \\ I_m \end{bmatrix}$$

and

$$\Lambda^{[\zeta]}(\cdot) = \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix} + \sigma^{-2} \begin{bmatrix} -CL_B^{[\zeta]} \\ \sigma B' \end{bmatrix} \left[ N_\sigma^{[\zeta]} \right]' (I - A' - \zeta I)^{-1} \begin{bmatrix} C' & L_C^{[\zeta]} B \end{bmatrix}$$

is such that  $K^{[\zeta]}(\cdot) \in H_{\infty, [1]}^c(\mathbb{C}^{\mathbb{P} \times \mathbb{m}})$  and  $\|G(i \cdot) + K^{[\zeta]}(i \cdot)\|_\infty \leq \sigma$ .

**Proof** This is a consequence of Theorem 5.2.3 and Theorem 6.2.4.  $\blacksquare$

**Example 3.1.2** (continued) Example 3.1.2 in Chapter 3 fits in both classes A and B considered above:

1.  $A - BB^*$ ,  $B$  and  $C := B^*$  are all bounded, and so  $\Sigma(A - BB^*, B, C)$  is a smooth Pritchard-Salamon system with  $W = V = \ell_2(\mathbb{N})$ .
2. Since  $A - BB^*$  is bounded, it generates the analytic semigroup  $\{e^{A - BB^* t}\}_{t \geq 0}$ . Moreover,  $B \in \mathcal{L}(\mathbb{C}^m, Z_\alpha)$  with  $m = 1$  and  $\alpha = 0$ .
3. Since  $A + A^* = 0$ , it follows from Corollary 2.2.3 (page 33, Curtain and Zwart [27]) that  $A$  generates a contraction semigroup. Thus from Lemma 2.2.6 (page 19, Oostveen [61]), it follows that the system given by the triple  $(A - BB^*, B, B^*)$  is output stable, that is, the map  $z \mapsto B^* e^{(A - BB^*)z} \in \mathcal{L}(\ell_2(\mathbb{Z}), L_2(0, \infty))$ . So in particular, with  $z = B \in \ell_2(\mathbb{Z})$ , we obtain  $h(\cdot) = B^* e^{(A - BB^*)B} \in L_2(0, \infty)$ . Furthermore, we have already verified in Chapter 3 that  $G(i \cdot)$  is continuous with the unique limit 1 at  $\pm\infty$ .

So the results in this section apply, solving the sub-optimal Hankel norm approximation problem for this non-exponentially stable example.  $\diamond$

# Chapter 7

## The case of regular linear systems

In this chapter, we will solve the sub-optimal Hankel norm approximation problem for regular linear systems with generating operators  $(A, B, C)$  satisfying some mild assumptions. In the special case of the sub-optimal Nehari problem, an explicit parameterization of all solutions is obtained in terms of the system parameters  $A$ ,  $B$  and  $C$ .

The limiting factor to extending the results from Chapter 5 to the general case of well-posed linear systems is the difficulty in manipulating with the unbounded operators  $B$  and  $C$ . In general, it is not clear that the candidate spectral factor given by (5.1) is even well-posed (see Staffans [78]).

In this chapter we translate the problem to the analogous one for reciprocal systems which we define in Section 7.1. In Section 7.2, we will solve the sub-optimal Hankel norm approximation problem for regular linear systems. The sub-optimal Nehari problem is a special case in which  $\sigma > \sigma_1$ . In this special case, using the results from Curtain and Oostveen [16], we can give a parameterization of *all* solutions, and we do this in the last section of this chapter.

### 7.1 Reciprocal systems

The *reciprocal system* of the regular linear system with generating operators  $(A, B, C)$  such that  $0 \in \rho(A)$  (here  $\rho(A)$  denotes the resolvent set of  $A$ ), is the regular linear system with the bounded generating operators  $(A^{-1}, A^{-1}B, -CA^{-1})$ . The key to our approach is the relationship between

the transfer function of the original regular linear system  $G(s) = C_\Lambda(sI - A)^{-1}B$  and that of its reciprocal system with the transfer function  $G_r(s) = -CA^{-1}(sI - A^{-1})^{-1}A^{-1}B$ . Note that  $A^{-1}$ ,  $A^{-1}B$ ,  $CA^{-1}$  are all bounded operators and for all  $s \in \rho(A) \cap \rho(A^{-1})$  there holds

$$C_\Lambda(sI - A)^{-1}B = -C_\Lambda A^{-1}B - CA^{-1} \left( \frac{1}{s} - A^{-1} \right)^{-1} A^{-1}B, \quad (7.1)$$

$$\begin{aligned} G(s) &= G_r \left( \frac{1}{s} \right) - C_\Lambda A^{-1}B \\ &= G_r \left( \frac{1}{s} \right) + G(0). \end{aligned} \quad (7.2)$$

From (7.2), it is clear that  $G \in H_\infty(\mathcal{L}(U, Y))$  iff  $G_r \in H_\infty(\mathcal{L}(U, Y))$ . In addition, the reciprocal system has the same controllability and observability Gramians, which we prove in the following lemma.

**Lemma 7.1.1** *Let  $(A, B, C)$  be generating operators of a regular linear system with transfer function  $G$ . Suppose that  $0 \in \rho(A)$  and  $G_r$  is the transfer function of its reciprocal system with generating operators  $(A^{-1}, A^{-1}B, -CA^{-1})$ . Then the following hold:*

1.  $C$  is an infinite-time admissible observation operator for  $A$  iff  $-CA^{-1}$  is an infinite-time admissible observation operator for  $A^{-1}$ . If either  $C$  or  $-CA^{-1}$  is infinite-time admissible (for the semigroups generated by  $A$  or  $A^{-1}$ , respectively), then the observability Gramians are identical.
2.  $B$  is an infinite-time admissible control operator for  $A$  iff  $A^{-1}B$  is an infinite-time admissible observation operator for  $A^{-1}$ . If either  $C$  or  $A^{-1}B$  is infinite-time admissible (for the semigroups generated by  $A$  or  $A^{-1}$ , respectively), then the observability Gramians are identical.
3.  $G \in H_\infty(\mathcal{L}(U, Y))$  iff  $G_r \in H_\infty(\mathcal{L}(U, Y))$ .

**Proof** 1. From Hansen and Weiss [45] (see also Grabowski [41]), we know that  $C$  is an infinite-time admissible observation operator iff the Lyapunov equation

$$\langle Az_1, L_C z_2 \rangle + \langle L_C z_1, Az_2 \rangle = -\langle Cz_1, Cz_2 \rangle \quad (7.3)$$

for all  $z_1$  and  $z_2$  in  $D(A)$ , has a nonnegative definite solution  $L_C = L_C^* \geq 0$ . The equation (7.3) is clearly equivalent to the Lyapunov equation

$$\langle x_1, L_C A^{-1} x_2 \rangle + \langle L_C A^{-1} x_1, x_2 \rangle = -\langle CA^{-1} x_1, CA^{-1} x_2 \rangle \quad (7.4)$$

for all  $x_1$  and  $x_2$  in  $X$ , which establishes the equivalence. Moreover, the observability Gramians are the smallest positive solution and so the Gramians are identical.

2. This is dual to part 1 above.

3. This follows from (7.2). ■

The idea is then to translate the sub-optimal Hankel norm approximation problem for the regular linear system with transfer function  $G$  into one for the system with transfer function  $G_{\mathbf{r}}$ , the latter having *bounded* generating operators. We now elaborate on this.

Clearly,  $K(-\cdot) \in H_{\infty,1}(\mathbb{C}^{\mathbf{p} \times \mathbf{m}})$  iff  $K_{\mathbf{r}}(-\cdot) \in H_{\infty,1}(\mathbb{C}^{\mathbf{p} \times \mathbf{m}})$ , where  $K$  and  $K_{\mathbf{r}}$  are related by

$$K_{\mathbf{r}}(s) = K\left(\frac{1}{s}\right) + G(0), \quad \text{for all } s \in \mathbb{C}_0^+.$$

Consequently we have  $\|G(i\cdot) + K(i\cdot)\|_{\infty} = \|G_{\mathbf{r}}(i\cdot) + K_{\mathbf{r}}(i\cdot)\|_{\infty}$ . Thus from Theorem 1.2.1, it follows that  $\sigma_{\mathbf{k}}(G) = \sigma_{\mathbf{k}}(G_{\mathbf{r}})$  for all  $\mathbf{k} \in \mathbb{N}$ .

## 7.2 The sub-optimal Hankel norm approximation problem

In this section we will solve the sub-optimal Hankel norm approximation problem for regular linear systems, by first translating the problem to its reciprocal system with *bounded* generating operators, albeit a *nonexponentially* stable semigroup. Subsequently we use Theorem 6.3.1 from Chapter 6 in order to obtain explicit formulae for solutions to the sub-optimal Hankel norm approximation problem for such systems.

We will solve the sub-optimal Hankel norm approximation problem for the regular linear system on a Hilbert space  $X$  with generating operators  $(A, B, C)$  under the following assumptions:

- ‡1.  $U = \mathbb{C}^{\mathbf{m}}$  and  $Y = \mathbb{C}^{\mathbf{p}}$ .
- ‡2.  $0 \in \rho(A)$  and  $\sigma(A) \cap \mathbb{C}_0^+$  is empty.
- ‡3.  $B$  is an infinite-time admissible control operator for  $\{T(t)\}_{t \geq 0}$ .
- ‡4.  $C$  is an infinite-time admissible observation operator for  $\{T(t)\}_{t \geq 0}$ .
- ‡5.  $G(\cdot) = C_{\Lambda}(\cdot I - A)^{-1}B \in H_{\infty}(\mathbb{C}^{\mathbf{p} \times \mathbf{m}})$ ,  $G(i\cdot) \in \mathcal{C}(\mathbb{R}, \mathbb{C}^{\mathbf{p} \times \mathbf{m}})$  and it has a unique limit at  $\pm\infty$ .

First of all, we will show that under if the assumptions  $\mathfrak{H}1 - \mathfrak{H}5$  are satisfied by the original system  $(A, B, C)$ , then the assumptions 1 – 3 of Class A on page 115 are satisfied by the reciprocal system  $(A^{-1}, A^{-1}B, -CA^{-1})$ . Furthermore, from (7.2) and Theorem 3.0.11, the Hankel operator of the original system  $H_G$  and that of the reciprocal system  $H_{G_r}$  are both compact. Also, as discussed in Section 7.1, it follows from Theorem 1.2.1 that the Hankel singular values of  $G$  and  $G_r$  are equal. Applying Theorem 6.3.1, we give solutions to the sub-optimal Hankel norm approximation problem for the reciprocal system and hence also to the original system. But first we will prove the following elementary result.

**Lemma 7.2.1** *If  $(A, B, C)$  satisfy  $\mathfrak{H}1 - \mathfrak{H}5$ , then the reciprocal system  $(A^{-1}, A^{-1}B, -CA^{-1})$  satisfies assumptions<sup>1</sup> 1 – 3 of Class A.*

**Proof** 1 is obvious.

2 : Owing to the infinite-time input admissibility of  $-CA^{-1}$  for the reciprocal system, it is clear that for all  $x \in X$ ,  $-CA^{-1}T(\cdot)x \in L_2([0, \infty), \mathbb{C}^p)$ . Consequently,  $-CA^{-1}T(\cdot)A^{-1}B \in L_2([0, \infty), \mathbb{C}^{p \times m})$ . Furthermore, the continuity of  $G_r$  on the imaginary axis follows from equation (7.2), and  $\mathfrak{H}5$ .

3 : It follows from  $\mathfrak{H}2$  and the spectral mapping theorem for the resolvent (see 1.13.(i), page 243, Engel and Nagel [31]) that  $\sigma(A^{-1}) \cap \mathbb{C}_0^+$  is empty. Furthermore, since  $A^{-1}$  is bounded, it satisfies the spectrum determined growth assumption (see for instance Corollary 2.4, page 252, Engel and Nagel [31]). Thus it is clear that the growth bound of the semigroup generated by  $A^{-1}$  is nonpositive, and so  $A^{-1} - \zeta I$  is the infinitesimal generator of an exponentially stable, strongly continuous semigroup. ■

We end this section by giving our main result.

**Theorem 7.2.2** *Suppose that  $(A, B, C)$  satisfies  $\mathfrak{H}1 - \mathfrak{H}5$ . Let  $\sigma_{1+1} < \sigma < \sigma_1$  and  $Q(\cdot) \in H_\infty(\mathbb{C}^{p \times m})$ , and  $\|Q(i \cdot)\|_\infty \leq 1$ . Then there exists a  $\delta > 0$  such that for every  $\zeta$  satisfying  $0 < \zeta < \delta$ ,*

$$K_r(-s) := R_1(-s)R_2(-s)^{-1},$$

where

$$\begin{bmatrix} R_1(-s) \\ R_2(-s) \end{bmatrix} = \Lambda(-s)^{-1} \begin{bmatrix} Q(-s - \zeta) \\ I_m \end{bmatrix},$$

$$\Lambda(s) = \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix} + \frac{\begin{bmatrix} -\frac{1}{\sigma^2}CA^{-1}L_B^{[\zeta]} \\ \frac{1}{\sigma}(A^{-1}B)^* \end{bmatrix} \left[ N_\sigma^{[\zeta]} \right]^* \left[ sI + (A^{-1})^* - \zeta I \right]^{-1} \left[ (CA^{-1})^* L_G^{[\zeta]} A^{-1}B \right]}{1}$$

<sup>1</sup>see page 115

is such that  $K_r(\cdot) \in H_{\infty,1}(\mathbb{C}^{\times m})$ . Here  $L_B^{[\zeta]}$  and  $L_C^{[\zeta]}$  denote the controllability Gramian and the observability Gramian, respectively, of the exponentially stable system  $(A^{-1} - \zeta I, A^{-1}B, -CA^{-1})$ , and  $N_\sigma^{[\zeta]} := \left[ I - \frac{1}{\sigma^2} L_B^{[\zeta]} L_C^{[\zeta]} \right]^{-1}$ . Furthermore, defining

$$K(s) = K_r \left( \frac{1}{s} \right) - G(0), \tag{7.5}$$

we have  $K(\cdot) \in H_{\infty,1}(\mathbb{C}^{\times m})$  and  $\|G(i\cdot) + K(i\cdot)\|_\infty \leq \sigma$ .

**Proof** From Lemma 7.2.1 it follows that the reciprocal system  $(A^{-1}, A^{-1}B, -CA^{-1})$  satisfies the assumptions 1 – 3 of Class A. Let  $G_r$  denote the transfer function of  $(A^{-1}, A^{-1}B, -CA^{-1})$ . Moreover, we have

$$\sigma_1(G_r) = \sigma_1(G) > \sigma > \sigma_{1+1}(G) = \sigma_{1+1}(G_r).$$

Consequently, using Theorem 6.3.1, we have that  $K_r(\cdot) \in H_{\infty,1}(\mathbb{C}^{\times m})$ <sup>2</sup> and  $\|G_r(i\cdot) + K_r(i\cdot)\|_\infty \leq \sigma$ . Finally it is clear that  $K$  defined by (7.5) is such that  $K(\cdot) \in H_{\infty,1}(\mathbb{C}^{\times m})$  and  $\|G(i\cdot) + K(i\cdot)\|_\infty = \|G_r(i\cdot) + K_r(i\cdot)\|_\infty \leq \sigma$ . ■

### 7.3 The sub-optimal Nehari problem

Finally, in this section we solve the sub-optimal Nehari problem for the regular linear system on a Hilbert space  $X$  with generating operators  $(A, B, C)$  under the following assumptions:

- ℳ1.  $U = \mathbb{C}^m$  and  $Y = \mathbb{C}$ .
- ℳ2.  $0 \in \rho(A)$ .
- ℳ3.  $B$  is an infinite-time admissible control operator for  $\{T(t)\}_{t \geq 0}$ .
- ℳ4.  $C$  is an infinite-time admissible observation operator for  $\{T(t)\}_{t \geq 0}$ .
- ℳ5.  $G(\cdot) = C_\Lambda(\cdot I - A)^{-1}B \in H_\infty(\mathbb{C}^{\times m})$ .

The sub-optimal Nehari problem can be thought of as a special case of the sub-optimal Hankel norm approximation problem with  $1 = 0$ . So in principle, the results of the previous section apply to this case. However, we do not do

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<sup>2</sup>Note that from Theorem 6.3.1, it follows that  $K_r(\cdot) \in H_{\infty,1}^c(\mathbb{C}^{\times m})$ , but this implies that  $K_r(\cdot) \in H_{\infty,1}(\mathbb{C}^{\times m})$

that in this section. This is because the Theorem 6.3.1, which was used in the previous section, can be now replaced (in the *special* case of the Nehari problem) by a more powerful result from Curtain and Oostveen [16], which will enable us to even obtain a *parameterization* of *all* solutions to the sub-optimal Nehari problem. Also, we find that with this alternate approach we can solve the problem under weaker assumptions than those demanded in the previous section (notice the differences in  $\mathfrak{N}2$  versus  $\mathfrak{H}2$ , and  $\mathfrak{N}5$  versus  $\mathfrak{H}5$ ). However, the broad approach in both sections remains the same: instead of looking at the original system, we translate the problem to the reciprocal system, solve it, and finally retrieve solutions to the original problem.

Suppose that  $\sigma > \sigma_1(G) = \sigma_1(G_r)$ . If  $K(\cdot) \in H_\infty(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$  satisfies  $\|G(i\cdot) + K(i\cdot)\|_\infty < \sigma$ , then  $K_r(\cdot) \in H_\infty(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$  satisfies  $\|G_r(i\cdot) + K_r(i\cdot)\|_\infty < \sigma$ , where

$$K_r(s) := K\left(\frac{1}{s}\right) + G(0).$$

Conversely, if  $K_r(\cdot) \in H_\infty(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$  satisfies  $\|G_r(i\cdot) + K_r(i\cdot)\|_\infty < \sigma$ , then  $K(\cdot) \in H_\infty(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$  satisfies  $\|G(i\cdot) + K(i\cdot)\|_\infty < \sigma$ , where

$$K(s) := K_r\left(\frac{1}{s}\right) - G(0).$$

So instead of solving the suboptimal Nehari problem for  $G$ , we solve the sub-optimal Nehari problem for the reciprocal system with the bounded generating operators  $A^{-1}$ ,  $A^{-1}B$ ,  $-CA^{-1}$ . This system satisfies all the conditions in the following result from Curtain and Oostveen [16], which we now recall.

**Theorem 7.3.1** *Suppose that the triple  $(A, B, C)$  satisfies the following assumptions:*

- T1.  $A$  is the infinitesimal generator of a strongly continuous semigroup on the Hilbert space  $X$ ,  $B \in \mathcal{L}(\mathbb{C}^{\mathfrak{m}}, X)$  and  $C \in \mathcal{L}(X, \mathbb{C}^{\mathfrak{p}})$ .
- T2.  $G(i\cdot) = C(\cdot iI - A)^{-1}B \in L_\infty(\mathbb{R}, \mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$ .
- T3.  $B$  is an infinite-time admissible control operator for  $\{T(t)\}_{t \geq 0}$ .
- T4.  $C$  is an infinite-time admissible observation operator for  $\{T(t)\}_{t \geq 0}$ .

Let  $\sigma > \sigma_1$  and let  $\Lambda$  be given by

$$\Lambda(s) = \begin{bmatrix} I_{\mathfrak{p}} & 0 \\ 0 & \sigma I_{\mathfrak{m}} \end{bmatrix} + \begin{bmatrix} -\frac{1}{\sigma^2}CL_B \\ \frac{1}{\sigma}B^* \end{bmatrix} N_\sigma^* [sI + A^*]^{-1} \begin{bmatrix} C^* & L_C B \end{bmatrix}.$$

where  $L_B$  and  $L_C$  denote the controllability Gramian and the observability Gramian, respectively, of the system  $(A, B, C)$ , and  $N_\sigma := [I - \frac{1}{\sigma^2}L_B L_C]^{-1}$ .

Then  $K(\cdot) \in H_\infty(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$  satisfies  $\|G(i) + K(i)\|_\infty \leq \sigma$  iff  $K(-s) = R_1(-s)R_2(-s)^{-1}$ , where

$$\begin{bmatrix} R_1(-s) \\ R_2(-s) \end{bmatrix} = \Lambda(-s)^{-1} \begin{bmatrix} Q(-s) \\ I_m \end{bmatrix}$$

for some  $Q(-s) \in H_\infty(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$  satisfying  $\|Q(i)\|_\infty \leq 1$ .

Indeed, under the assumptions  $\mathfrak{N}1 - \mathfrak{N}5$ , the reciprocal system has its transfer function in  $H_\infty(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$  and the operators  $A^{-1}B$  and  $-CA^{-1}$  are infinite-time admissible with the same Gramians  $L_B$  and  $L_C$  as the original system. So in light of what has been said above, upon applying Theorem 7.3.1 to the reciprocal system  $(A^{-1}, A^{-1}B, -CA^{-1})$ , we obtain the following result:

**Theorem 7.3.2** *Suppose that  $(A, B, C)$  satisfies  $\mathfrak{N}1 - \mathfrak{N}5$ . Let  $\sigma > \sigma_1$  and  $Q(\cdot) \in H_\infty(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$ , and  $\|Q(i)\|_\infty \leq 1$ . Let  $\Lambda$  be given by*

$$\Lambda(s) = \begin{bmatrix} I_p & 0 \\ 0 & \sigma I_m \end{bmatrix} + \begin{bmatrix} -\frac{1}{\sigma^2}CA^{-1}L_B \\ \frac{1}{\sigma}(A^{-1}B)^* \end{bmatrix} N_\sigma^* \left[ sI + (A^{-1})^* \right]^{-1} \left[ (CA^{-1})^* L_C A^{-1}B \right],$$

where  $L_B$  and  $L_C$  denote the controllability Gramian and the observability Gramian, respectively, of the system  $(A, B, C)$ , and  $N_\sigma := \left[ I - \frac{1}{\sigma^2}L_B L_C \right]^{-1}$ . Then  $K(\cdot) \in H_\infty(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$  satisfies  $\|G(i) + K(i)\|_\infty \leq \sigma$  iff

$$K(s) = K_r \left( \frac{1}{s} \right) + G(0),$$

where  $K_r(-s) = R_1(-s)R_2(-s)^{-1}$ , and

$$\begin{bmatrix} R_1(-s) \\ R_2(-s) \end{bmatrix} = \Lambda(-s)^{-1} \begin{bmatrix} Q(-s) \\ I_m \end{bmatrix},$$

for some  $Q(-s) \in H_\infty(\mathbb{C}^{\mathfrak{p} \times \mathfrak{m}})$  satisfying  $\|Q(i)\|_\infty \leq 1$ .

While it is tempting to try to write  $\Lambda$  and  $K$  in terms of its reciprocal, we know that this will not (in general) be well-defined (see Staffans [78]). So we leave the explicit solution as it stands.

# Chapter 8

## Coda

In this last chapter, we discuss the approach followed in this book to solve the sub-optimal Hankel norm approximation problem for infinite-dimensional systems.

In Chapter 4, we gave sufficient frequency domain conditions to solve the sub-optimal Hankel norm approximation problem. Subsequently in Chapter 5, we solved the sub-optimal Hankel norm approximation problem for the exponentially stable smooth Pritchard-Salamon class and the class of exponentially stable analytic systems, by explicitly verifying the frequency domain conditions given in Chapter 4. In Chapter 6, under certain conditions, we gave a solution to the sub-optimal Hankel norm approximation problem in the case when the infinitesimal generator is not necessarily exponentially stable and finally, in Chapter 7 we solved the sub-optimal Hankel norm approximation problem in the case of regular linear systems.

The main advantage of our approach in this book is that it is a self-contained, elementary solution to the sub-optimal Hankel norm approximation problem. Moreover, our results in Chapter 4 hold for a class larger than the Wiener class. Although our results in Chapter 4 hold for a class larger than the Wiener class, unfortunately, except for the results in Chapters 6 and 7, most of our examples have  $h \in L_1$ !

One might conjecture that an approach similar to the one in this book would yield solutions to the sub-optimal Hankel norm approximation problem for more general classes of well-posed linear systems than the ones considered in this book. However, we suspect this task to be a formidable one, since we need the smoothness properties of the controllability and observability Gramians to verify the candidate spectral factor formulas given in this book.

This book also gives alternate state-space formulae for the sub-optimal Hankel norm approximation problem for finite-dimensional systems. We conclude this book by explicitly working out an elementary finite-dimensional example, which elucidates the procedure involved and the theorems in Sections 5.1 and 5.2 of Chapter 5.

**Example 8.0.3** (*An elementary finite-dimensional example.*) Consider the rational function

$$G(s) = \frac{1}{(s + \frac{1}{2})^2},$$

which has the minimal (and hence controllable) state-space realization of MacMillan degree 2 given by

$$A = \begin{bmatrix} -\frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad C = [1 \quad 0].$$

The impulse response is  $h(t) = Ce^{tA}B = te^{-\frac{1}{2}t}$  for  $t \geq 0$  and its controllability and observability Gramians are given by

$$L_B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad L_C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},$$

respectively. We have

$$L_B L_C = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}, \quad \sigma(L_B L_C) = \{3 + 2\sqrt{2}, 3 - 2\sqrt{2}\},$$

and so the Hankel singular values of the system are

$$\sigma_1 = \sqrt{2} + 1 > \sigma_2 = \sqrt{2} - 1.$$

Let  $\mathbf{1} = 1$  and  $\sigma = 1 \in (\sqrt{2} - 1, \sqrt{2} + 1)$ . Thus we seek a sub-optimal Hankel norm approximant of  $G$  of MacMillan degree at most 1. We have

$$\begin{aligned} V(s) &= \begin{bmatrix} I_p & 0 \\ 0 & \frac{1}{\sigma} I_m \end{bmatrix} - \frac{1}{\sigma^2} \begin{bmatrix} -CL_B \\ B' \end{bmatrix} (sI + A')^{-1} N'_\sigma [C' \quad \frac{1}{\sigma} L_C B] \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s - \frac{1}{2} & 0 \\ 1 & s - \frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 - \frac{\frac{1}{2}}{(s - \frac{1}{2})^2} & -\frac{s-1}{(s - \frac{1}{2})^2} \\ \frac{s}{(s - \frac{1}{2})^2} & 1 - \frac{\frac{1}{2}}{(s - \frac{1}{2})^2} \end{bmatrix}. \end{aligned}$$

With  $Q(-s) = 0$ , we obtain

$$K_0(-s) = V_{12}(-s)V_{22}(-s)^{-1}$$

$$\begin{aligned}
&= \frac{s+1}{\left(s+\frac{1}{2}\right)^2} \cdot \left[1 - \frac{\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2}\right]^{-1} \\
&= \frac{s+1}{\left[s+\frac{1}{2}-\frac{1}{\sqrt{2}}\right]\left[s+\frac{1}{2}+\frac{1}{\sqrt{2}}\right]} \\
&= \frac{\frac{1}{2}+\frac{1}{2\sqrt{2}}}{s+\frac{1}{2}-\frac{1}{\sqrt{2}}} + \frac{\frac{1}{2}-\frac{1}{2\sqrt{2}}}{s+\frac{1}{2}+\frac{1}{\sqrt{2}}}.
\end{aligned}$$

Thus one sub-optimal Hankel norm approximant of  $G$  is

$$G_0(s) = \frac{\frac{1}{2} + \frac{1}{2\sqrt{2}}}{s - \frac{1}{2} + \frac{1}{\sqrt{2}}},$$

and  $\|G - G_0\|_H \leq 1$ .

◇

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# Standard notation

Here we list some standard notation that we use in the thesis. Non standard symbols which are defined in the thesis can be looked up in the Index.

## Set theory

1.  $\mathbb{N}$ : the set of natural numbers  $\{1, 2, 3, \dots\}$ .
2.  $\complement S$ : complement of the set  $S$ .

## Linear algebra

Throughout this thesis, unless otherwise specified, when considering a vector space, the underlying field is  $\mathbb{C}$ . In a matrix, blank entries are always to be understood to be zeros.

1.  $\dim(\mathcal{V})$ , where  $\mathcal{V}$  is a finite-dimensional vector space: The dimension of  $\mathcal{V}$ .
2.  $\ker(T)$  and  $\text{ran}(T)$ , where  $T : \mathcal{V}_1 \rightarrow \mathcal{V}_2$  is a linear transformation from the vector space  $\mathcal{V}_1$  to the vector space  $\mathcal{V}_2$ : the null space and the range of  $T$ , respectively.

## Linear analysis

1.  $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ , where  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are Hilbert spaces: the Banach space of all bounded linear operators with the norm  $\|T\| = \sup_{(0 \neq) x \in \mathcal{H}_1} \frac{\|Tx\|_{\mathcal{H}_2}}{\|x\|_{\mathcal{H}_1}}$ .
2.  $\rho(A)$ ,  $\sigma(A)$ ,  $\sigma_p(A)$ ,  $\sigma_c(A)$ , where  $A : D(A) (\subset \mathcal{H}) \rightarrow \mathcal{H}$  is a closed linear operator, and  $\mathcal{H}$  is a Hilbert space: the resolvent set, spectrum, point spectrum and continuous spectrum, respectively, of  $A$ . If  $D(A)$  is dense in  $\mathcal{H}$ , then  $A^*$  denotes the adjoint of  $A$ .

3.  $\mathcal{C}(X_1, X_2)$ , where  $X_1$  and  $X_2$  are topological spaces: the space of continuous maps from  $X_1$  to  $X_2$ . If  $X_2$  is  $\mathbb{C}$ , we simply write  $\mathcal{C}(X_1)$  instead of  $\mathcal{C}(X_1, \mathbb{C})$ .
4.  $\ell_2(\mathbb{N})$  ( $\ell_2(\mathbb{Z})$ ): the Hilbert space of square summable complex sequences indexed by  $\mathbb{N}$  (respectively  $\mathbb{Z}$ ), with the inner product

$$\langle a, b \rangle_2 = \sum_{n=1}^{\infty} a_n \overline{b_n} \quad \left( \text{respectively } \langle a, b \rangle_2 = \sum_{n=-\infty}^{\infty} a_n \overline{b_n} \right).$$

5.  $\ell_1$ : the Banach space of summable complex sequences indexed by  $\mathbb{N}$  with the norm

$$\|a\|_1 = \sum_{n=1}^{\infty} |a_n|.$$

6.  $\ell_{\infty}$ : the Banach space of complex sequences indexed by  $\mathbb{N}$  and bounded in absolute value with the norm

$$\|a\|_{\infty} = \sup_{n \in \mathbb{N}} |a_n|.$$

7.  $L_2^{\text{loc}}(\Omega, X)$ , where  $\Omega \subset \mathbb{R}$  is measurable and  $X$  is a Banach space:

$$\left\{ f : \Omega \rightarrow X \mid \begin{array}{l} f \text{ is Bochner-Lebesgue measurable and} \\ \int_K \|f(\omega)\|^2 d\omega < \infty \text{ for any compact set } K \subset \Omega \end{array} \right\}.$$

$L_p(\Omega, X)$ , where  $1 \leq p < \infty$ :

$$\left\{ f : \Omega \rightarrow X \mid \begin{array}{l} f \text{ is Bochner-Lebesgue measurable and} \\ \int_{\Omega} \|f(\omega)\|^p d\omega < \infty \end{array} \right\}$$

with the norm  $\|f\|_p = \left( \int_{\Omega} \|f(\omega)\|^p d\omega \right)^{\frac{1}{p}}$ . If  $X$  is  $\mathbb{C}$ , we simply write  $L_p(\Omega)$  instead of  $L_p(\Omega, \mathbb{C})$ .

If  $p = 2$  and  $X$  is a Hilbert space, then it is also a Hilbert space with the inner product  $\langle f, g \rangle_2 = \int_{\Omega} \langle f(\omega), g(\omega) \rangle d\omega$ .

$L_{\infty}(\Omega, X)$ :

$$\left\{ f : \Omega \rightarrow X \mid \begin{array}{l} f \text{ is Bochner-Lebesgue measurable and} \\ \text{esssup}_{\omega \in \Omega} \|f(\omega)\| < \infty \end{array} \right\}$$

with the norm  $\|f\|_{\infty} = \text{esssup}_{\omega \in \Omega} \|f(\omega)\|$ .

8.  $H_{\infty}(X)$ , where  $X$  is a Banach space:

$$\left\{ f : \mathbb{C}_0^+ \rightarrow X \mid f \text{ is analytic and } \sup_{s \in \mathbb{C}_0^+} \|f(s)\| < \infty \right\}.$$